# Factorial ANOVA

More than one categorical explanatory variable

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# Factorial ANOVA

- Categorical explanatory variables are called **factors**
- More than one at a time
- Primarily for true experiments, but also used with observational data

- If there are observations at all combinations of explanatory variable values, it's called a *complete* factorial design (as opposed to a fractional factorial).

# The potato study

- Cases are storage containers (of potatoes)
- Same number of potatoes in each container. Inoculate with bacteria, store for a fixed time period.
- Response variable is number of rotten potatoes.
- Two explanatory variables, randomly assigned
  - Bacteria Type (1, 2, 3)
  - Temperature (1=Cool, 2=Warm)

# Two-factor design

| | **Bacteria Type** | | |
|---|---|---|---|
| **Temp** | 1 | 2 | 3 |
| 1=Cool | | | |
| 2=Warm | | | |

Six treatment conditions

# Factorial experiments

- Allow more than one factor to be investigated in the same study: Efficiency!
- Allow the scientist to see whether the effect of an explanatory variable *depends* on the value of another explanatory variable: Interactions
- Thank you again, Mr. Fisher.

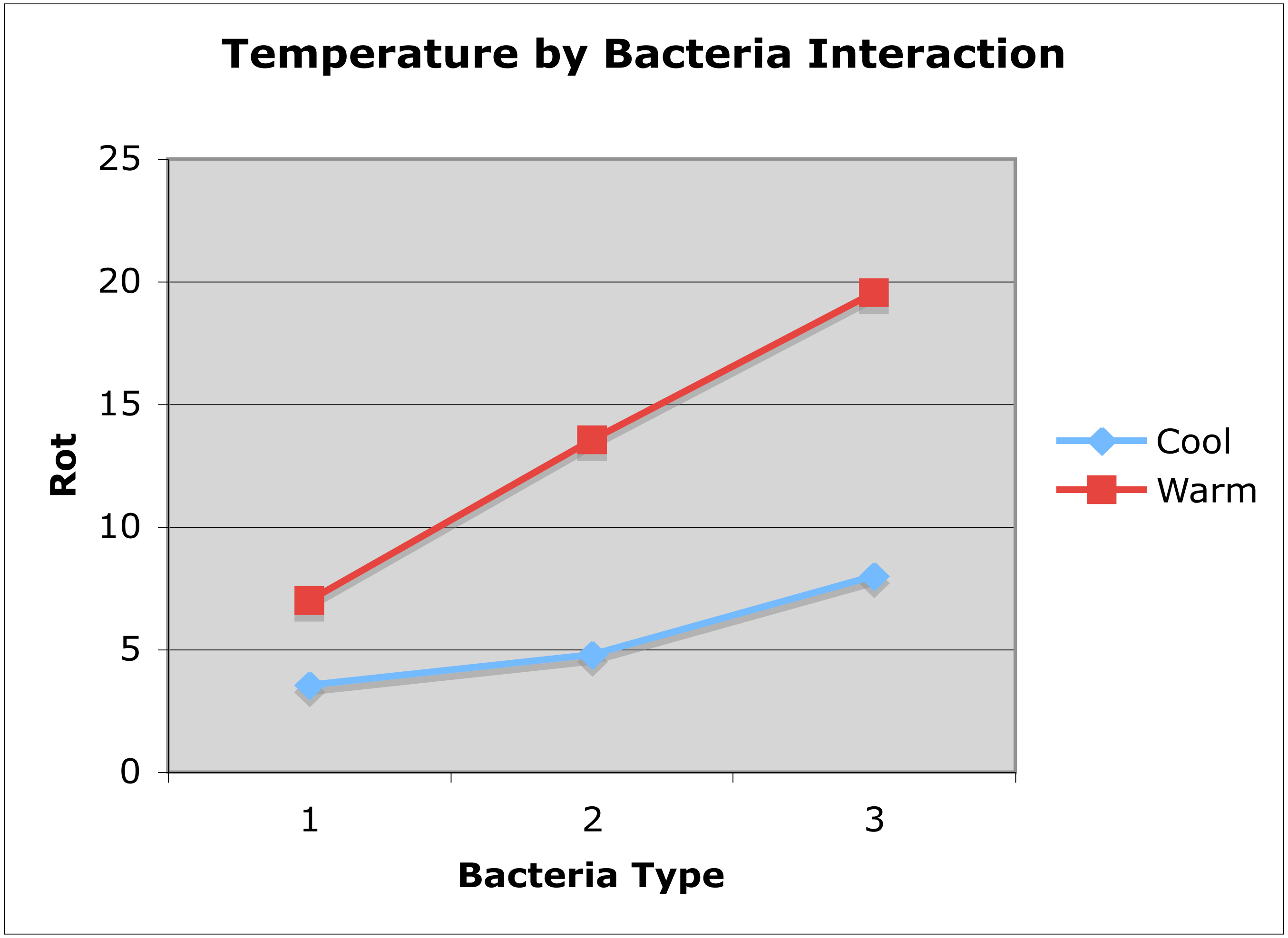
# Normal with equal variance and conditional (cell) means $\mu_{i,j}$

| | **Bacteria Type** | | | |
|---|---|---|---|---|
| **Temp** | 1 | 2 | 3 | |
| 1=Cool | $\mu_{1,1}$ | $\mu_{1,2}$ | $\mu_{1,3}$ | $\frac{\mu_{1,1}+\mu_{1,2}+\mu_{1,3}}{3}$ |
| 2=Warm | $\mu_{2,1}$ | $\mu_{2,2}$ | $\mu_{2,3}$ | $\frac{\mu_{2,1}+\mu_{2,2}+\mu_{2,3}}{3}$ |
| | $\frac{\mu_{1,1}+\mu_{2,1}}{2}$ | $\frac{\mu_{1,2}+\mu_{2,2}}{2}$ | $\frac{\mu_{1,3}+\mu_{2,3}}{2}$ | $\mu$ |

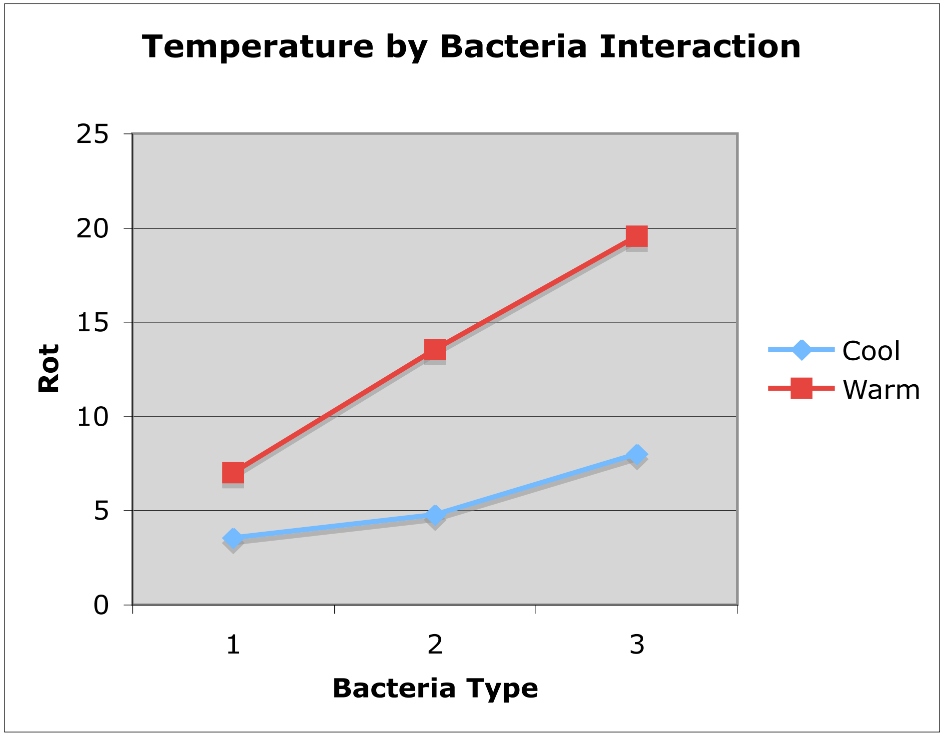
# Tests

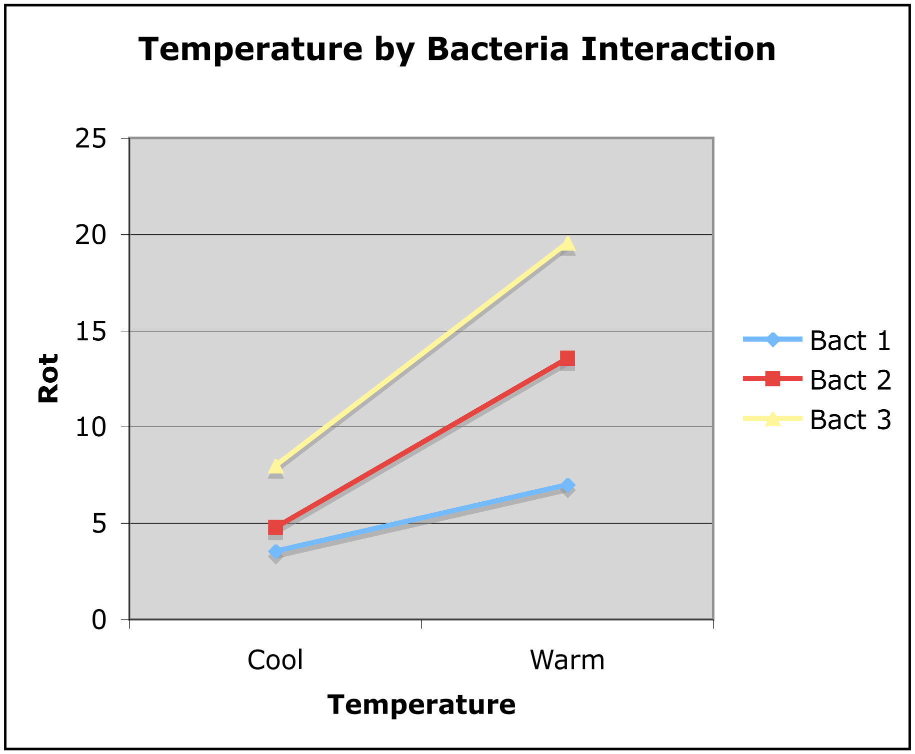
- Main effects: Differences among marginal means
- Interactions: Differences between differences (What is the effect of Factor A? **It depends** on the level of Factor B.)

# To understand the interaction, plot the means

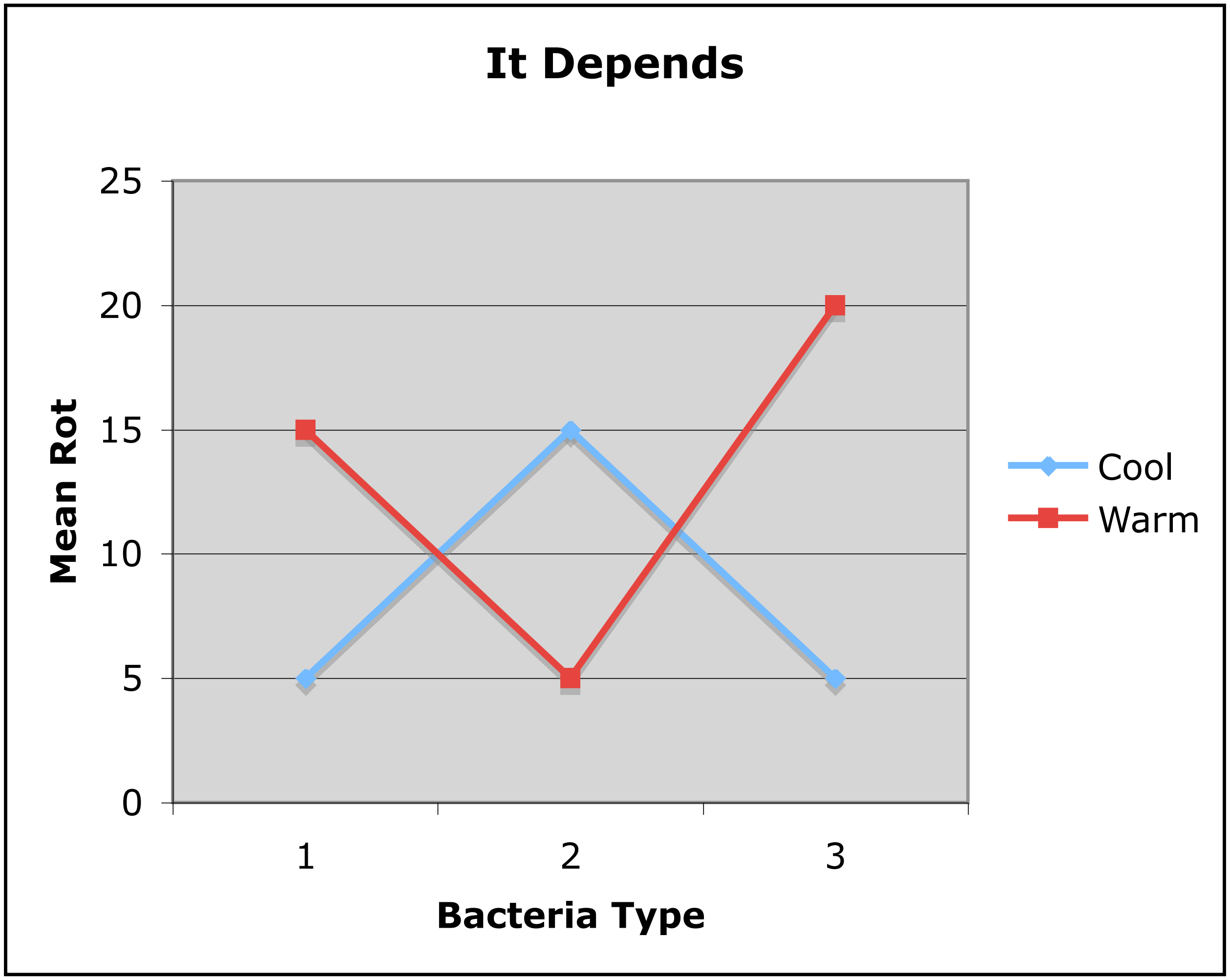


# Either Way

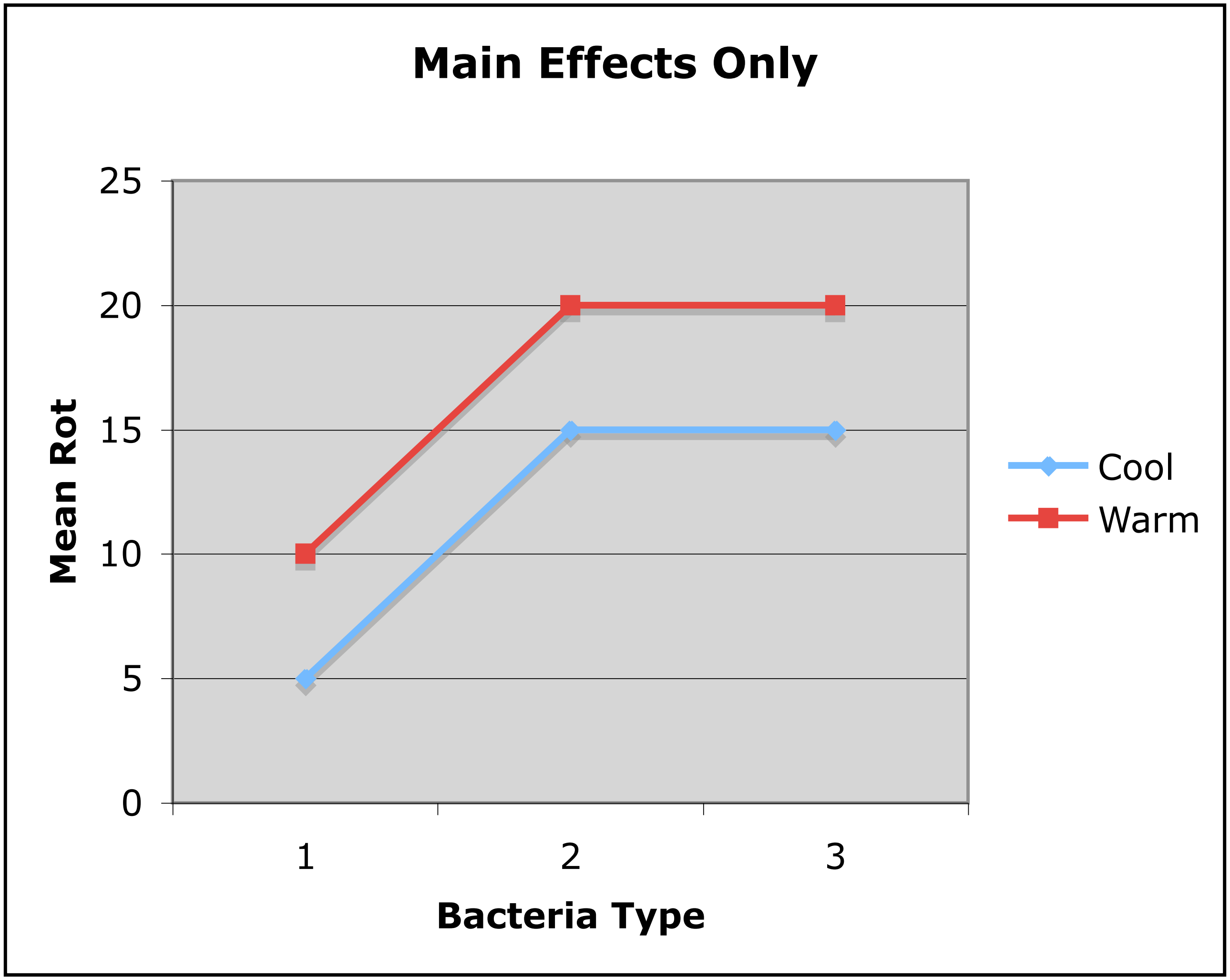

Temperature by Bacteria Interaction
25
20
15
10
5
0
Rot
1
2
3
Bacteria Type
Cool
Warm


Temperature by Bacteria Interaction
25
20
15
10
5
0
Rot
Cool
Warm
Temperature
Bact 1
Bact 2
Bact 3

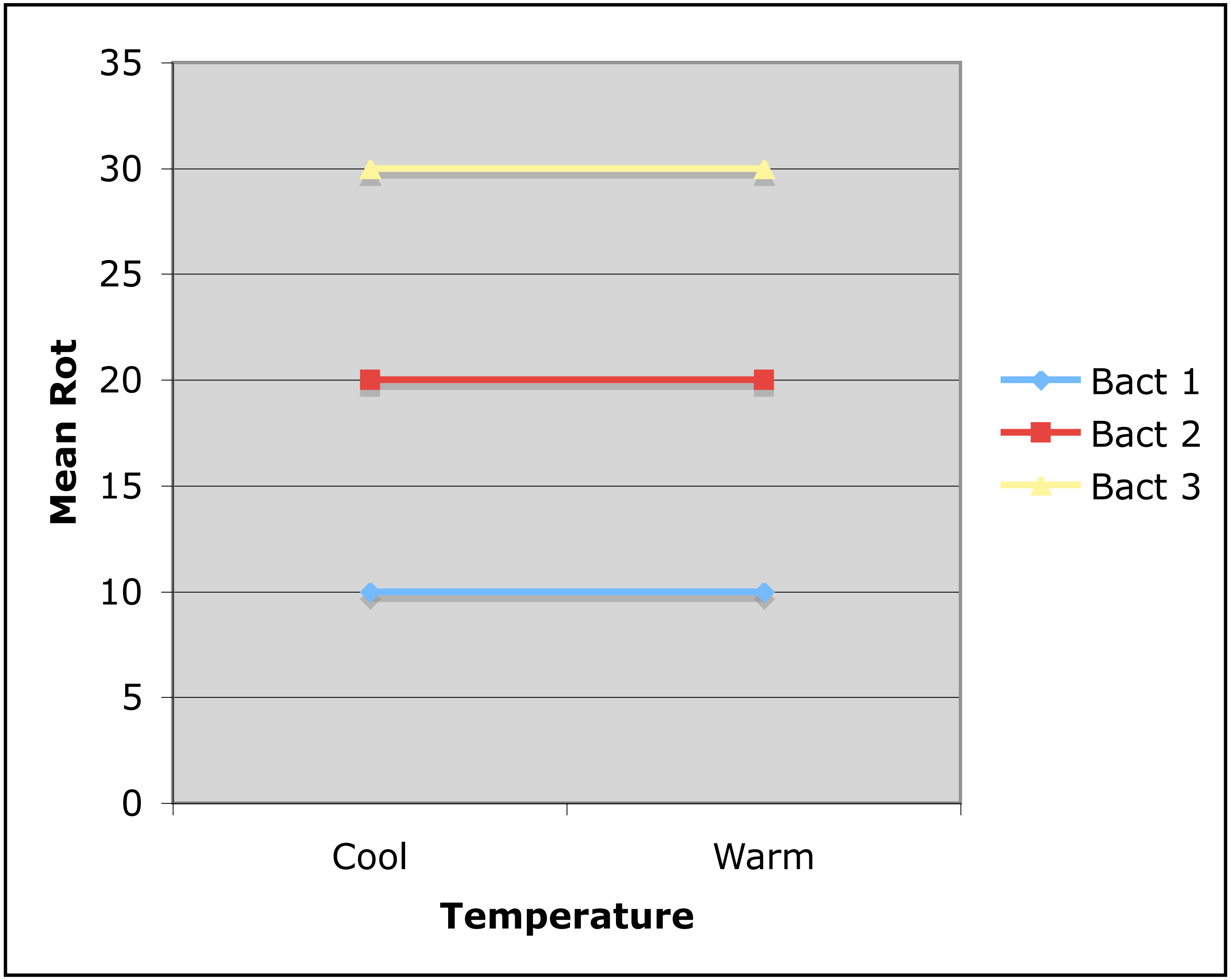
# Non-parallel profiles = Interaction



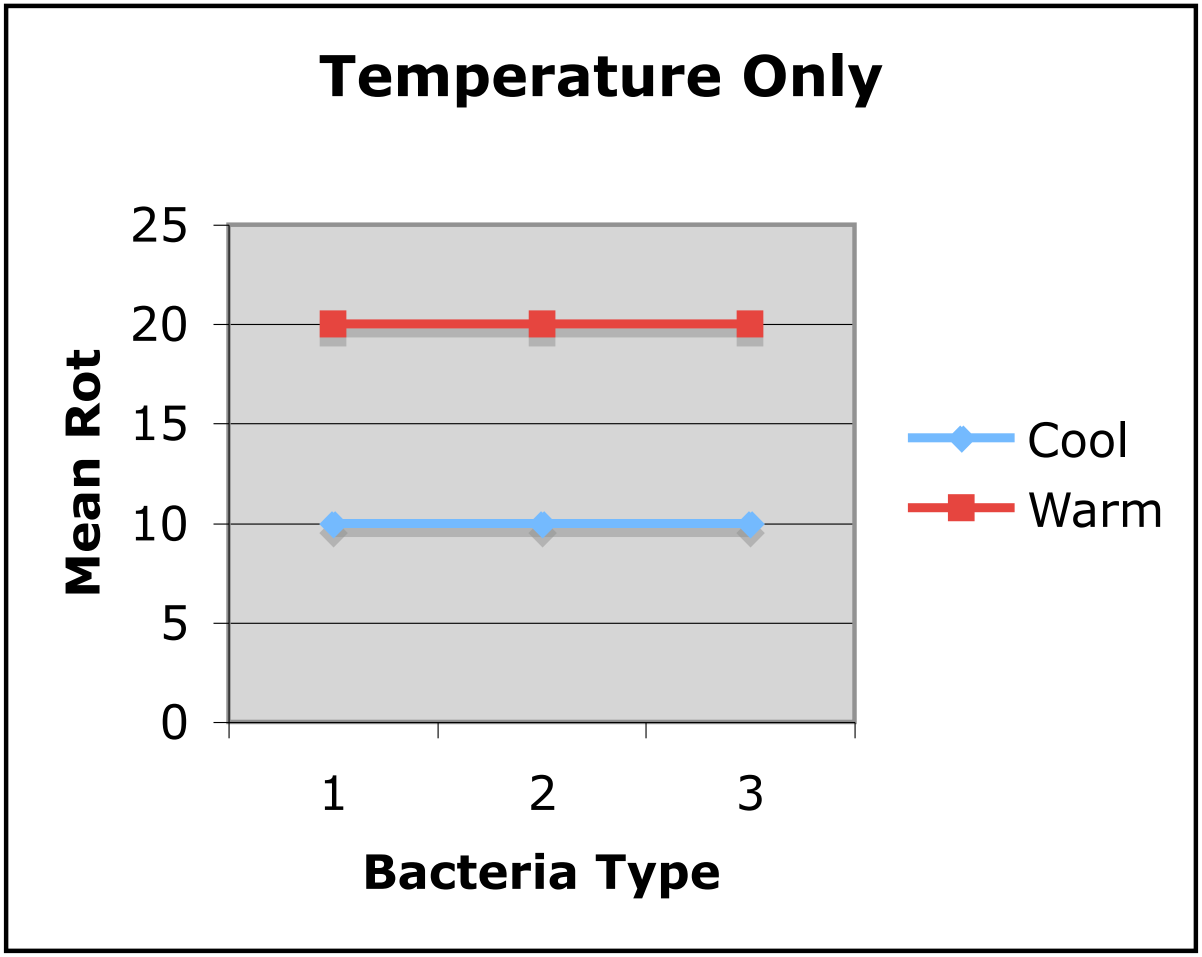
# Main effects for both variables, no interaction



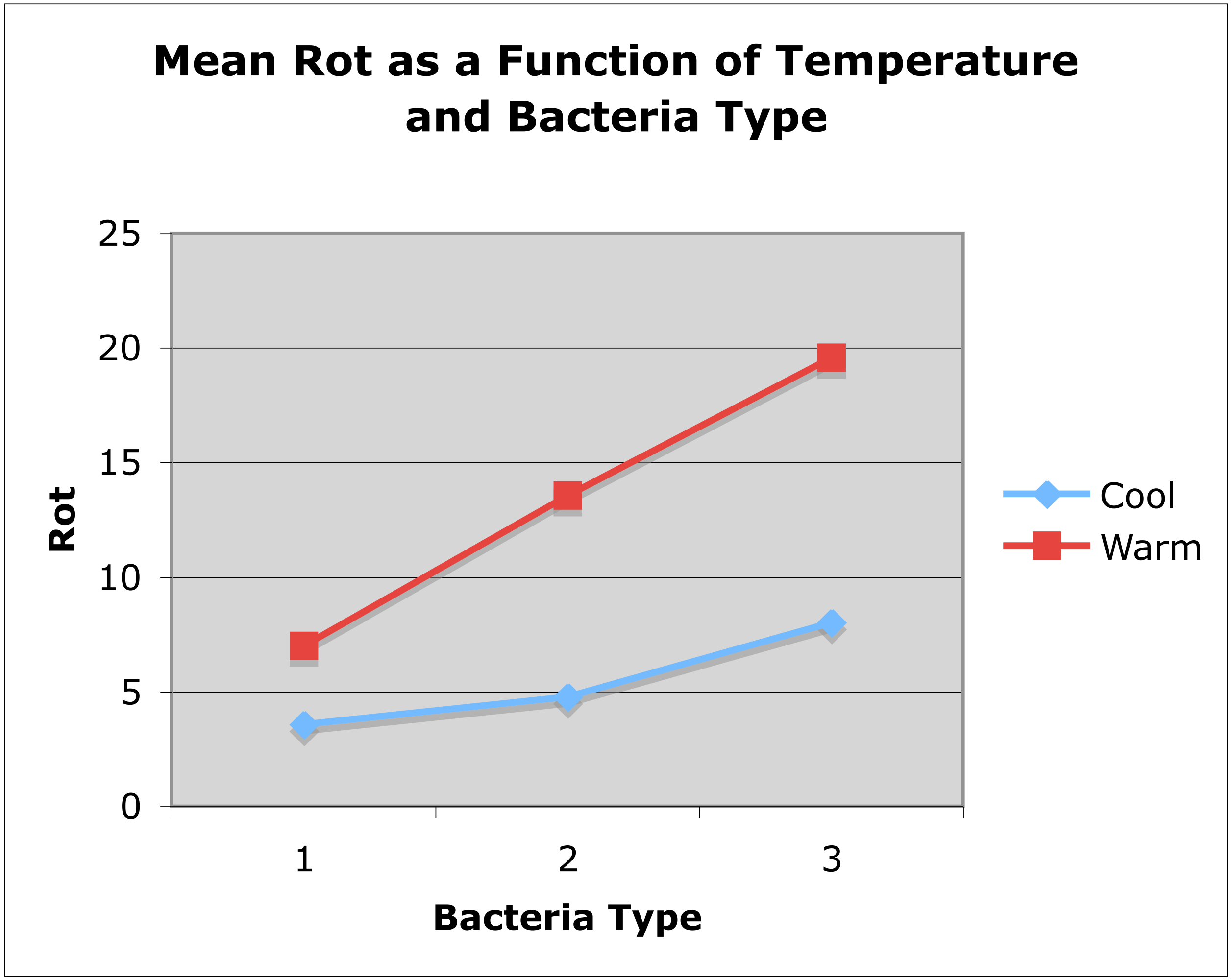
# Main effect for Bacteria only



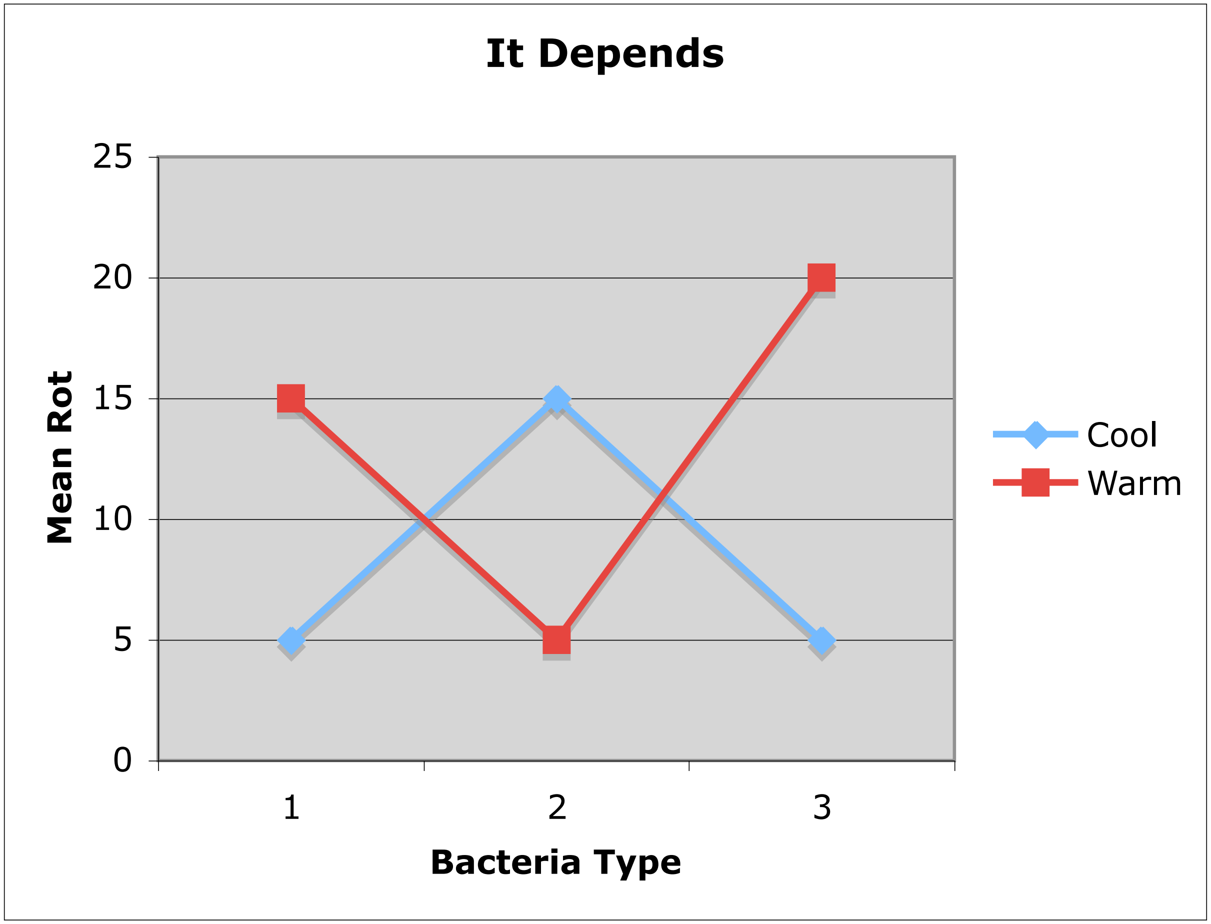
# Main Effect for Temperature Only



# Both Main Effects, and the Interaction



# Should you interpret the main effects?



# Testing Contrasts

| | Bacteria Type | | | |
|---|---|---|---|---|
| **Temp** | 1 | 2 | 3 | |
| 1=Cool | $\mu_{1,1}$ | $\mu_{1,2}$ | $\mu_{1,3}$ | $\frac{\mu_{1,1}+\mu_{1,2}+\mu_{1,3}}{3}$ |
| 2=Warm | $\mu_{2,1}$ | $\mu_{2,2}$ | $\mu_{2,3}$ | $\frac{\mu_{2,1}+\mu_{2,2}+\mu_{2,3}}{3}$ |
| | $\frac{\mu_{1,1}+\mu_{2,1}}{2}$ | $\frac{\mu_{1,2}+\mu_{2,2}}{2}$ | $\frac{\mu_{1,3}+\mu_{2,3}}{2}$ | $\mu$ |

- Differences between marginal means are definitely contrasts
- Interactions are also sets of contrasts

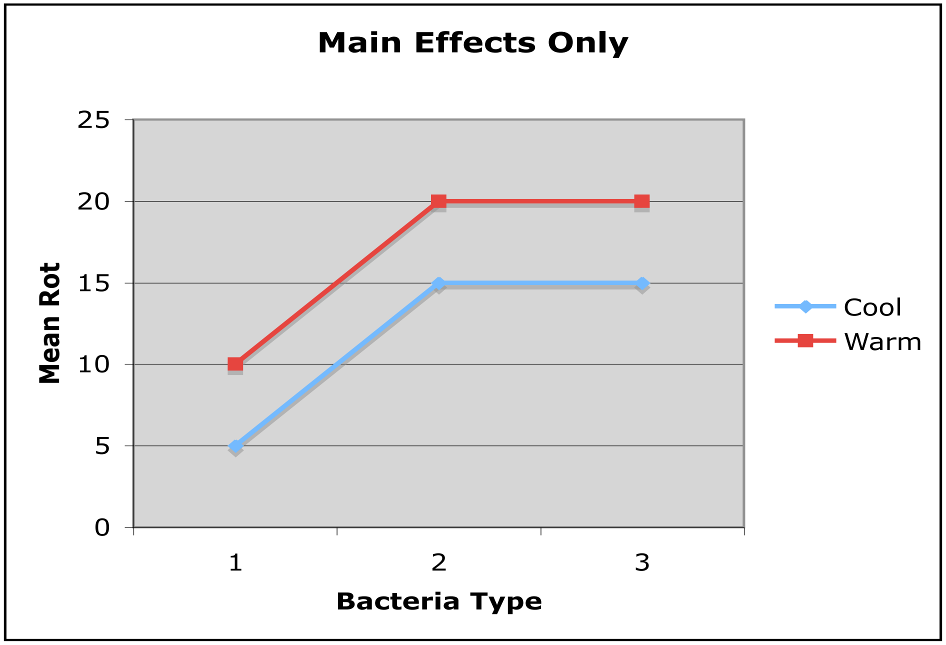
# Interactions are sets of Contrasts

| | Bacteria Type | | | |
|---|---|---|---|---|
| **Temp** | 1 | 2 | 3 | |
| 1=Cool | $\mu_{1,1}$ | $\mu_{1,2}$ | $\mu_{1,3}$ | $\frac{\mu_{1,1}+\mu_{1,2}+\mu_{1,3}}{3}$ |
| 2=Warm | $\mu_{2,1}$ | $\mu_{2,2}$ | $\mu_{2,3}$ | $\frac{\mu_{2,1}+\mu_{2,2}+\mu_{2,3}}{3}$ |
| | $\frac{\mu_{1,1}+\mu_{2,1}}{2}$ | $\frac{\mu_{1,2}+\mu_{2,2}}{2}$ | $\frac{\mu_{1,3}+\mu_{2,3}}{2}$ | $\mu$ |

- $H_0 : \mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3}$

- $H_0 : \mu_{1,2} - \mu_{1,1} = \mu_{2,2} - \mu_{2,1}$ and
  $\mu_{1,3} - \mu_{1,2} = \mu_{2,3} - \mu_{2,2}$

# Interactions are sets of Contrasts




- $H_0 : \mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3}$

- $H_0 : \mu_{1,2} - \mu_{1,1} = \mu_{2,2} - \mu_{2,1}$ and

  $\mu_{1,3} - \mu_{1,2} = \mu_{2,3} - \mu_{2,2}$

# Equivalent statements

- The effect of A depends upon B
- The effect of B depends on A

$$H_0 : \mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3}$$

$$H_0 : \mu_{1,2} - \mu_{1,1} = \mu_{2,2} - \mu_{2,1} \text{ and}$$
$$\mu_{1,3} - \mu_{1,2} = \mu_{2,3} - \mu_{2,2}$$

# Three factors: A, B and C

- There are three (sets of) main effects: One each for A, B, C
- There are three two-factor interactions
  - A by B (Averaging over C)
  - A by C (Averaging over B)
  - B by C (Averaging over A)
- There is one three-factor interaction: AxBxC

# Meaning of the 3-factor interaction

- The form of the A x B interaction depends on the value of C
- The form of the A x C interaction depends on the value of B
- The form of the B x C interaction depends on the value of A
- These statements are equivalent. Use the one that is easiest to understand.

# To graph a three-factor interaction

- Make a separate mean plot (showing a 2-factor interaction) for each value of the third variable.

- In the potato study, a graph for each type of potato

# Four-factor design

- Four sets of main effects
- Six two-factor interactions
- Four three-factor interactions
- One four-factor interaction: The nature of the three-factor interaction depends on the value of the 4th factor
- There is an F test for each one
- And so on …

# As the number of factors increases

- The higher-way interactions get harder and harder to understand
- All the tests are still tests of sets of contrasts (differences between differences of differences …)
- But it gets harder and harder to write down the contrasts
- Effect coding becomes easier

# Effect coding

| Bact | $B_1$ | $B_2$ |
|---|---|---|
| 1 | 1 | 0 |
| 2 | 0 | 1 |
| 3 | -1 | -1 |

| Temperature | T |
|---|---|
| 1=Cool | 1 |
| 2=Warm | -1 |

$$E(Y|\mathbf{X}=\mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

# Interaction effects are products of dummy variables

$$E(Y|\mathbf{X} = \mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

- The A x B interaction: Multiply each dummy variable for A by each dummy variable for B
- Use these products as additional explanatory variables in the multiple regression
- The A x B x C interaction: Multiply each dummy variable for C by each product term from the A x B interaction
- Test the sets of product terms simultaneously

# Make a table

$$E(Y|\mathbf{X}=\mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

| Bact | Temp | $B_1$ | $B_2$ | T | $B_1$T | $B_2$T | $E(Y\|\mathbf{X}=\mathbf{x})$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 | 1 | 0 | $\beta_0 + \beta_1 + \beta_3 + \beta_4$ |
| 1 | 2 | 1 | 0 | -1 | -1 | 0 | $\beta_0 + \beta_1 - \beta_3 - \beta_4$ |
| 2 | 1 | 0 | 1 | 1 | 0 | 1 | $\beta_0 + \beta_2 + \beta_3 + \beta_5$ |
| 2 | 2 | 0 | 1 | -1 | 0 | -1 | $\beta_0 + \beta_2 - \beta_3 - \beta_5$ |
| 3 | 1 | -1 | -1 | 1 | -1 | -1 | $\beta_0 - \beta_1 - \beta_2 + \beta_3 - \beta_4 - \beta_5$ |
| 3 | 2 | -1 | -1 | -1 | 1 | 1 | $\beta_0 - \beta_1 - \beta_2 - \beta_3 + \beta_4 + \beta_5$ |

# Cell and Marginal Means

| | **Bacteria Type** | | | |
|---|---|---|---|---|
| **Tmp** | 1 | 2 | 3 | |
| 1=C | $\beta_0 + \beta_1 + \beta_3 + \beta_4$ | $\beta_0 + \beta_2 + \beta_3 + \beta_5$ | $\beta_0 - \beta_1 - \beta_2$ $+\beta_3 - \beta_4 - \beta_5$ | $\beta_0$ $+\beta_3$ |
| 2=W | $\beta_0 + \beta_1 - \beta_3 - \beta_4$ | $\beta_0 + \beta_2 - \beta_3 - \beta_5$ | $\beta_0 - \beta_1 - \beta_2$ $-\beta_3 + \beta_4 + \beta_5$ | $\beta_0$ $-\beta_3$ |
| | $\beta_0 + \beta_1$ | $\beta_0 + \beta_2$ | $\beta_0 - \beta_1 - \beta_2$ | $\beta_0$ |

# We see

- Intercept is the grand mean
- Regression coefficients for the dummy variables are deviations of the marginal means from the grand mean
- What about the interactions?

$$E(Y|\mathbf{X}=\mathbf{x}) = \beta_0 + \beta_1 B_1 + \beta_2 B_2 + \beta_3 T + \beta_4 B_1 T + \beta_5 B_2 T$$

# A bit of algebra shows

$\mu_{1,1} - \mu_{2,1} = \mu_{1,2} - \mu_{2,2}$ is equivalent to $\beta_4 = \beta_5$

$\mu_{1,2} - \mu_{2,2} = \mu_{1,3} - \mu_{2,3}$ is equivalent to $\beta_4 = -\beta_5$

So $\beta_4 = \beta_5 = 0$

# Factorial ANOVA with effect coding is pretty automatic

- You don’t have to make a table unless asked
- It always works as you expect it will
- Hypothesis tests are the same as testing sets of contrasts
- Covariates present no problem. Main effects and interactions have their usual meanings, “controlling” for the covariates.
- Plot the “least squares means” (Y-hat at x-bar values for covariates)

# Again

- Intercept is the grand mean
- Regression coefficients for the dummy variables are deviations of the marginal means from the grand mean
- Test of main effect(s) is test of the dummy variables for a factor.
- Interaction effects are products of dummy variables.

# Balanced vs. Unbalanced Experimental Designs

- Balanced design: Cell sample sizes are proportional (maybe equal)
- Explanatory variables have zero relationship to one another
- Numerator SS in ANOVA are independent
- Everything is nice and simple
- Most experimental studies are designed this way.
- As soon as somebody drops a test tube, it's no longer true

# Analysis of unbalanced data

- When explanatory variables are related, there is potential ambiguity.
- A is related to Y, B is related to Y, and A is related to B.
- Who gets credit for the portion of variation in Y that could be explained by either A or B?
- With a regression approach, whether you use contrasts or dummy variables (equivalent), the answer is **nobody**.
- Think of full, reduced models.
- Equivalently, general linear test

# Some software is designed for balanced data

- The special purpose formulas are much simpler.
- Very useful *in the past*.
- Since most data are at least a little unbalanced, a recipe for trouble.
- Most textbook data are balanced, so they cannot tell you what your software is really doing.
- R's anova and aov functions are designed for balanced data, though anova applied to lm objects can give you what you want if you use it with care.

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