Wald-Like Tests

- Distribution free
- Basic idea is like a large-sample Z-test
- Example: $X_1, \ldots, X_n$ a random sample from a distribution with mean μ and variance σ^2
- $H_0: \mu = \mu_0$

$$Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu_0)}{\widehat{\sigma}_n}$$

- $W = Z^2$ is Chisquare(1)

- Suppose

$$\mathbf{Y}_n = \sqrt{n}(\mathbf{T}_n - \boldsymbol{\theta}) \stackrel{d}{\rightarrow} \mathbf{Y} \sim N_k(\mathbf{0}, \boldsymbol{\Sigma})$$

- And H_0: ***C*$\boldsymbol{\vartheta}$ = *h*** is true.
- Then asymptotically (approximately, for large *n*)

$$\sqrt{n}(\mathbf{CT}_n - \mathbf{h}) \sim N_r(\mathbf{0}, \mathbf{C\Sigma C}')$$

and

$$\begin{aligned} W &= \sqrt{n}(\mathbf{CT}_n - \mathbf{h})' \left(\mathbf{C}\widehat{\boldsymbol{\Sigma}}_n\mathbf{C}'\right)^{-1} \sqrt{n}(\mathbf{CT}_n - \mathbf{h}) \\ &= n(\mathbf{CT}_n - \mathbf{h})' \left(\mathbf{C}\widehat{\boldsymbol{\Sigma}}_n\mathbf{C}'\right)^{-1} (\mathbf{CT}_n - \mathbf{h}) \\ &\sim \chi^2(r) \end{aligned}$$

Can be made rigorous

$$\mathbf{Y}_n = \sqrt{n}(\mathbf{T}_n - \boldsymbol{\theta}) \stackrel{d}{\rightarrow} \mathbf{Y} \sim N_k(\mathbf{0}, \boldsymbol{\Sigma}) \qquad \widehat{\boldsymbol{\Sigma}}_n \stackrel{p}{\rightarrow} \boldsymbol{\Sigma}$$

By Slutsky 6a (continuous mapping), $\mathbf{CY}_n \stackrel{d}{\rightarrow} \mathbf{CY} \sim N_r(\mathbf{0}, \mathbf{C\Sigma C}')$

By Slutsky 7a (continuous mapping),

$$\left(\mathbf{C}\widehat{\boldsymbol{\Sigma}}_n\mathbf{C}'\right)^{-1} \stackrel{p}{\rightarrow} (\mathbf{C\Sigma C}')^{-1}$$

By Slutsky 6c,

$$\left(\begin{array}{c} \mathbf{CY}_n \\ \left(\mathbf{C}\widehat{\boldsymbol{\Sigma}}_n\mathbf{C}'\right)^{-1} \end{array} \right) \stackrel{d}{\rightarrow} \left(\begin{array}{c} \mathbf{CY} \\ (\mathbf{C\Sigma C}')^{-1} \end{array} \right)$$

And if $H_0 : \mathbf{C}\boldsymbol{\theta} = \mathbf{h}$ is true, by By Slutsky 6a (continuous mapping),

$$\begin{aligned} (\mathbf{CY}_n)' \left(\mathbf{C}\widehat{\boldsymbol{\Sigma}}_n\mathbf{C}'\right)^{-1} (\mathbf{CY}_n) &= n(\mathbf{CT}_n - \mathbf{h})' \left(\mathbf{C}\widehat{\boldsymbol{\Sigma}}_n\mathbf{C}'\right)^{-1} (\mathbf{CT}_n - \mathbf{h}) \\ &\stackrel{d}{\rightarrow} (\mathbf{CY})' (\mathbf{C\Sigma C}')^{-1} (\mathbf{CY}) \sim \chi^2(r) \end{aligned}$$

Example

- Customers can purchase a computer with up to 10 extra options, such as a bigger monitor, more RAM, larger hard drive, printer, etc.
- Options selected were recorded for a sample of 400 customers.
- Data are binary (Yes-No) but correlated.
- Data file looks like this

ID	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	0	0	1	0	1	1	0	0	1	0
2	0	0	0	0	0	1	0	0	0	1
3	1	1	0	0	1	0	0	0	1	0
					Etc.					

Want to Test

- Null hypothesis is all selection probabilities are equal
- If rejected, which ones are different from each other? (Pairwise comparisons)
- But the full 2x2x …x2 = 2^{10} = 1024-cell contingency table has too many parameters to estimate.
- However, the multivariate Central Limit Theorem applies, and the sample variance-covariance matrix is a consistent estimator of $\boldsymbol{\Sigma}$.

Independent groups (Between cases)

- Have n cases, separated into k groups: Maybe occupation of main wage earner in family
- $n_1 + n_2 + \ldots + n_k = n$
- Dependent variable is either binary or amount of something, like annual energy consumption
- No reason to believe normality
- No reason to believe equal variances
- H_0: $\mathbf{C\mu} = \mathbf{h}$
- For example, H_0: $\mu_1 = \ldots = \mu_k$

Basic Idea

The *k* group means are independent random variables. Asymptotically,

- $\overline{X}_j \sim N(\mu_j, \frac{\sigma_j^2}{n_j})$
- The $k \times 1$ random vector $\overline{\mathbf{X}}_n \sim N(\boldsymbol{\mu}, \mathbf{V})$,
- Where $\mathbf{V}$ is a $k \times k$ diagonal matrix with jth diagonal element $\frac{\sigma_j^2}{n_j}$.
- $\mathbf{C}\overline{\mathbf{X}}_n \sim N_r(\mathbf{C}\boldsymbol{\mu}, \mathbf{CVC}')$
- Approximate $\mathbf{V}$ with the diagonal matrix $\widehat{\mathbf{V}}$, jth diagonal element $\frac{\widehat{\sigma}_j^2}{n_j}$
- And if $H_0 : \mathbf{C}\boldsymbol{\mu} = \mathbf{h}$ is true,

$$W = (\mathbf{C}\overline{\mathbf{X}}_n - \mathbf{h})' \left(\mathbf{C}\widehat{\mathbf{V}}\mathbf{C}'\right)^{-1} (\mathbf{C}\overline{\mathbf{X}}_n - \mathbf{h}) \sim \chi^2(r)$$

One little technical issue

- More than one n_j is going to infinity
- The rates at which they go to infinity can't be too different
- In particular, if $n = n_1 + ... + n_k$
- Then each n_j/n must converge to a non-zero constant (in probability).