# STA 2201/442 Assignment 3

1. For each of the following distributions, derive a general expression for the Maximum Likelihood Estimator (MLE). Carry out the second derivative test to make sure you really have a maximum. Then use the data to calculate a numerical estimate; you should bring a calculator to the quiz in case you have to do something like this.

   (a) $p(x) = \theta(1-\theta)^x$ for $x = 0, 1, \ldots$, where $0 < \theta < 1$. Data: `4, 0, 1, 0, 1, 3, 2, 16, 3, 0, 4, 3, 6, 16, 0, 0, 1, 1, 6, 10`. Answer: 0.2061856

   (b) $f(x) = \frac{\alpha}{x^{\alpha+1}}$ for $x > 1$, where $\alpha > 0$. Data: `1.37, 2.89, 1.52, 1.77, 1.04, 2.71, 1.19, 1.13, 15.66, 1.43` Answer: 1.469102

   (c) $f(x) = \frac{\tau}{\sqrt{2\pi}} e^{-\frac{\tau^2 x^2}{2}}$, for $x$ real, where $\tau > 0$. Data: `1.45, 0.47, -3.33, 0.82, -1.59, -0.37, -1.56, -0.20` Answer: 0.6451059

   (d) $f(x) = \frac{1}{\theta} e^{-x/\theta}$ for $x > 0$, where $\theta > 0$. Data: `0.28, 1.72, 0.08, 1.22, 1.86, 0.62, 2.44, 2.48, 2.96` Answer: 1.517778

2. For each of the following distributions and associated data sets, obtain the maximum likelihood estimate numerically with `R`. Bring your printout for each problem to the quiz; you may be asked to hand it in. There are links to the data from the course web page in case the ones from this document do not work.

   (a) $f(x) = \frac{e^{x-\theta}}{(1+e^{x-\theta})^2}$ for $x$ real, where $-\infty < \theta < \infty$. Data:

```
4.82  3.66  4.39  1.66  3.80  4.69  1.73  4.50  9.29  4.05  4.50 -0.64  1.40
4.18  2.70  5.65  5.47  0.55  4.64  1.19  2.28  7.16  4.80  3.19  2.33  2.57
2.31  0.35  2.81  2.35  2.52  3.44  2.71 -1.43  7.61  0.93  2.52  6.86  6.14
4.37  3.79  5.04  4.50  1.92  3.25 -0.06  2.81  3.09  2.95  3.69
```

   You might want to read the data from `logistic.data`.

   (b) $f(x) = \frac{1}{\pi[1+(x-\theta)^2]}$ for $x$ real, where $-\infty < \theta < \infty$. Data:

```
3.77  3.57 -4.10 -4.87  4.18  4.59  5.27  8.33 -5.55  4.35  0.55 -5.57 34.78
-5.05 -2.18 -4.12  3.24 -3.78  3.57 -4.86
```

   You might want to read the data from `cauchy.data`. For this one, try at least two different starting values and *plot the minus log likelihood function!*

   (c) $f(x) = \frac{1}{2} e^{-|x-\theta|}$ for $x$ real, where $-\infty < \theta < \infty$. Data:

```
4.20  1.35  5.78  2.35  3.68  3.10  0.32  0.98  5.03  3.15  3.62  3.10  2.63
3.07  2.54  4.90  4.31  2.91  2.70  2.70  2.39  2.84  4.64  1.26  5.43  3.32
3.20  2.78  4.02  2.01  1.15  3.14  3.01 -1.03  1.13  0.14  1.52  2.94  2.80
5.04  2.61  4.51  2.17  2.62  3.16  3.99  3.29  2.69  3.26  2.85
```

   You might want to read the data from `dexp.data`.

(d) $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ for $0 < x < 1$, where $\alpha > 0$ and $\beta > 0$. Data:

```
0.45 0.42 0.38 0.26 0.43 0.24 0.32 0.50 0.44 0.29 0.45 0.29 0.29 0.32 0.30
0.32 0.30 0.38 0.43 0.35 0.32 0.33 0.29 0.20 0.46 0.31 0.35 0.27 0.29 0.46
0.43 0.37 0.32 0.28 0.20 0.26 0.39 0.35 0.35 0.24 0.36 0.28 0.32 0.23 0.25
0.43 0.30 0.43 0.33 0.37
```

You might want to read the data from `beta.data`. If you are getting a lot of warnings, maybe it's because the numerical search is leaving the parameter space. If so, try `help(nlminb)`.

(e) $f(x) = \frac{1}{m!}e^{-x}x^m$ for $x > 0$, where the unknown parameter $m$ is a positive integer. *This means your estimate will be an integer.* Data:

```
 8.34  7.65  6.72  3.84  7.12  1.88  5.07  2.69  4.50  5.78  4.88  5.23  6.17
11.76  7.84  5.87  5.23  6.55  8.34  5.35  4.98 13.81  8.62  7.88  6.34  5.16
 6.64  4.35  6.77  5.83  5.85  2.46  8.33  3.74  5.10  3.95  7.84  4.70  6.09
 5.23  1.44  6.11  4.88  7.24  7.89  8.98  1.78  5.46  5.34  4.25
```

You might want to read the data from `gamma.data`.

For each distribution, be able to state (briefly) why differentiating the log likelihood and setting the derivative to zero does not work. For the computer part, bring to the quiz one sheet of printed output for each distribution. The sheets should be separate, because you may hand only one of them in. Each printed page should show the following, *in this order*.

- Definition of the function that computes the likelihood, or log likelihood, or minus log likelihood or whatever.
- How you got the data into R – probably a `scan` statement.
- Listing of the data for the problem.
- The `nlm` or `nlminb` statement and resulting output.
- For the Cauchy example, a plot of the minus log likelihood.

3. For the data of Problem 2d, conduct a large-sample likelihood ratio test of $H_0 : \alpha = \beta$, using `R`. Your printout should display the value of $G^2$, the degrees of freedom and the $p$-value. Do you reject $H_0$ at the 0.05 significance level? If yes, which parameter seems to be higher based on $\widehat{\alpha}$ and $\widehat{\beta}$?

4. A fast food chain is considering a change in the blend of coffee beans they use to make their coffee. To determine whether their customers prefer the new blend, the company selects a random sample of coffee-drinking customers and asks them to taste coffee made with the old blend and two new alternatives, in cups marked "A," "B" and "C." Labels are in a different random order for each customer.

   (a) Derive a general formula for the (unrestricted) maximum likelihood estimator (MLE) of $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$. Show your work. In case you didn't realize it, this is a multinomial.

   (b) Suppose $n = 150$ consumers participate in the taste test. Thirty-eight prefer the old blend, 55 prefer new alternative One, and 57 prefer new alternative Two. Carry out a large-sample likelihood ratio test to determine whether there is a real difference in preference for the blends of coffee; use $\alpha = 0.05$. Use `R`. Your printout should display the value of $G^2$, the degrees of freedom and the $p$-value. Do you reject $H_0$ at the 0.05 significance level?