STA 2101/442 Assignment 1 (Review)

1. Which statement is true? (Quantities in boldface are matrices of constants.)
   (a) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
   (b) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{BA}+\mathbf{CA}$
   (c) Both a and b
   (d) Neither a nor b
2. Which statement is true?
   (a) $a(\mathbf{B}+\mathbf{C}) = a\mathbf{B}+a\mathbf{C}$
   (b) $a(\mathbf{B}+\mathbf{C}) = \mathbf{B}a+\mathbf{C}a$
   (c) Both a and b
   (d) Neither a nor b
3. Which statement is true?
   (a) $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{AB}+\mathbf{AC}$
   (b) $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$
   (c) Both a and b
   (d) Neither a nor b
4. Which statement is true?
   (a) $(\mathbf{AB})' = \mathbf{A}'\mathbf{B}'$
   (b) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
   (c) Both a and b
   (d) Neither a nor b
5. Which statement is true?
   (a) $\mathbf{A}'' = \mathbf{A}$
   (b) $\mathbf{A}''' = \mathbf{A}'$
   (c) Both a and b
   (d) Neither a nor b

6. Suppose that the square matrices $\mathbf{A}$ and $\mathbf{B}$ both have inverses. Which statement is true?

   (a) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$
   (b) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
   (c) Both a and b
   (d) Neither a nor b

7. Which statement is true?

   (a) $(\mathbf{A}+\mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
   (b) $(\mathbf{A}+\mathbf{B})' = \mathbf{B}' + \mathbf{A}'$
   (c) $(\mathbf{A}+\mathbf{B})' = (\mathbf{B}+\mathbf{A})'$
   (d) All of the above
   (e) None of the above

8. Which statement is true?

   (a) $(a+b)\mathbf{C} = a\mathbf{C} + b\mathbf{C}$
   (b) $(a+b)\mathbf{C} = \mathbf{C}a + \mathbf{C}b$
   (c) $(a+b)\mathbf{C} = \mathbf{C}(a+b)$
   (d) All of the above
   (e) None of the above

9. Recall that $\mathbf{A}$ symmetric means $\mathbf{A} = \mathbf{A}'$. Let $\mathbf{X}$ be an n by p matrix. Prove that $\mathbf{X}'\mathbf{X}$ is symmetric.

10. Recall that an inverse of the matrix $\mathbf{A}$ (denoted $\mathbf{A}^{-1}$) is defined by two properties: $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Prove that inverses are unique, as follows. Let $\mathbf{B}$ and $\mathbf{C}$ both be inverses of $\mathbf{A}$. Show that $\mathbf{B} = \mathbf{C}$.

11. Let $\mathbf{X}$ be an n by p matrix with $n \neq p$. Why is it incorrect to say that $(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{X}^{-1}\mathbf{X}'^{-1}$?

12. Suppose that the square matrices $\mathbf{A}$ and $\mathbf{B}$ both have inverses. Prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. You have two things to show.

13. Let $\mathbf{A}$ be a non-singular square matrix. Prove $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$.

14. Using Question 13, prove that the if the inverse of a symmetric matrix exists, it is also symmetric.

15. Let $\mathbf{A}$ be a square matrix with the determinant of $\mathbf{A}$ (denoted $|\mathbf{A}|$) equal to zero. What does this tell you about $\mathbf{A}^{-1}$? No proof is required here.

16. Let $\mathbf{a}$ be an $n \times 1$ matrix of real constants. How do you know $\mathbf{a}'\mathbf{a} \geq 0$?

17. Let $\mathbf{X}$ be an $n \times p$ matrix of constants. Recall the definition of linear independence. The columns of $\mathbf{X}$ are said to be *linearly dependent* if there exists a $p \times 1$ vector $\mathbf{v} \neq \mathbf{0}$ with $\mathbf{X}\mathbf{v} = \mathbf{0}$. We will say that the columns of $\mathbf{X}$ are linearly *independent* if $\mathbf{X}\mathbf{v} = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$.

    (a) Show that if the columns of $\mathbf{X}$ are linearly dependent, then the columns of $\mathbf{X}'\mathbf{X}$ are also linearly dependent.

    (b) Show that if the columns of $\mathbf{X}$ are linearly dependent, then the *rows* of $\mathbf{X}'\mathbf{X}$ are linearly dependent.

    (c) Show that if the columns of $\mathbf{X}$ are linearly independent, then the columns of $\mathbf{X}'\mathbf{X}$ are also linearly independent. Use Problem 16 and the definition of linear independence.

18. Let $\mathbf{A}$ be a square matrix. Show that if the columns of $\mathbf{A}$ are linearly dependent, $\mathbf{A}^{-1}$ cannot exist. Hint: $\mathbf{v}$ cannot be both zero and not zero at the same time.

19. Recall the *spectral decomposition* of a square symmetric matrix (For example, a variance-covariance matrix). Any such matrix $\boldsymbol{\Sigma}$ can be written as $\boldsymbol{\Sigma} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}'$, where $\mathbf{P}$ is a matrix whose columns are the (orthonormal) eigenvectors of $\boldsymbol{\Sigma}$, $\boldsymbol{\Lambda}$ is a diagonal matrix of the corresponding (non-negative) eigenvalues, and $\mathbf{P}'\mathbf{P} = \mathbf{P}\mathbf{P}' = \mathbf{I}$.

    (a) Let $\boldsymbol{\Sigma}$ be a square symmetric matrix with eigenvalues that are all strictly positive.

        i. What is $\boldsymbol{\Lambda}^{-1}$?

        ii. Show $\boldsymbol{\Sigma}^{-1} = \mathbf{P}\boldsymbol{\Lambda}^{-1}\mathbf{P}'$

    (b) Let $\boldsymbol{\Sigma}$ be a square symmetric matrix, and this time some of the eigenvalues might be zero.

        i. What do you think $\boldsymbol{\Lambda}^{1/2}$ might be?

        ii. Define $\boldsymbol{\Sigma}^{1/2}$ as $\mathbf{P}\boldsymbol{\Lambda}^{1/2}\mathbf{P}'$. Show $\boldsymbol{\Sigma}^{1/2}$ is symmetric.

        iii. Show $\boldsymbol{\Sigma}^{1/2}\boldsymbol{\Sigma}^{1/2} = \boldsymbol{\Sigma}$.

(c) Now return to the situation where the eigenvalues of the square symmetric matrix $\mathbf{\Sigma}$ are all strictly positive. Define $\mathbf{\Sigma}^{-1/2}$ as $\mathbf{P}\mathbf{\Lambda}^{-1/2}\mathbf{P}'$, where the elements of the diagonal matrix $\mathbf{\Lambda}^{-1/2}$ are the reciprocals of the corresponding elements of $\mathbf{\Lambda}^{1/2}$.

   i. Show that the inverse of $\mathbf{\Sigma}^{1/2}$ is $\mathbf{\Sigma}^{-1/2}$, justifying the notation.

   ii. Show $\mathbf{\Sigma}^{-1/2}\mathbf{\Sigma}^{-1/2} = \mathbf{\Sigma}^{-1}$.

(d) The (square) matrix $\mathbf{\Sigma}$ is said to be *positive definite* if $\mathbf{a}'\mathbf{\Sigma}\mathbf{a} > 0$ for all vectors $\mathbf{a} \neq \mathbf{0}$. Show that the eigenvalues of a symmetric positive definite matrix are all strictly positive. Hint: the **a** you want is an eigenvector.

(e) Let $\mathbf{\Sigma}$ be a symmetric, positive definite matrix. Putting together a couple of results you have proved above, establish that $\mathbf{\Sigma}^{-1}$ exists.