# Introduction to Time Series[1]

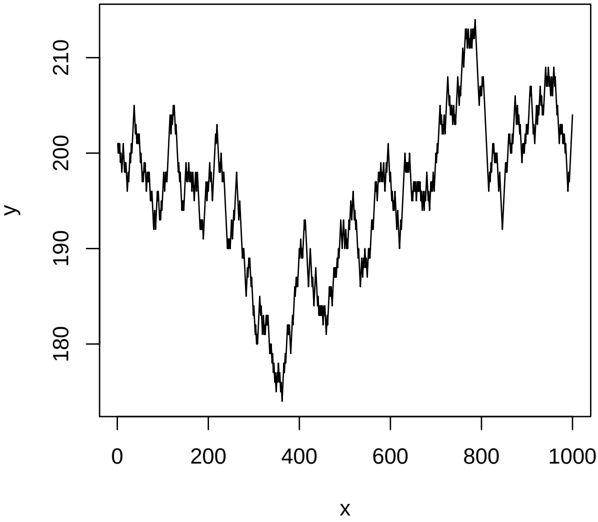
[1] This slide show is an open-source document. See last slide for copyright information.

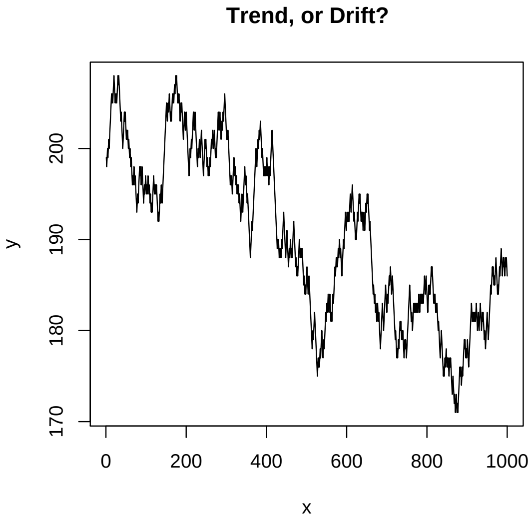
# Time series

A sequence of measurements (random variables) $X_1, X_2, \ldots, X_n$

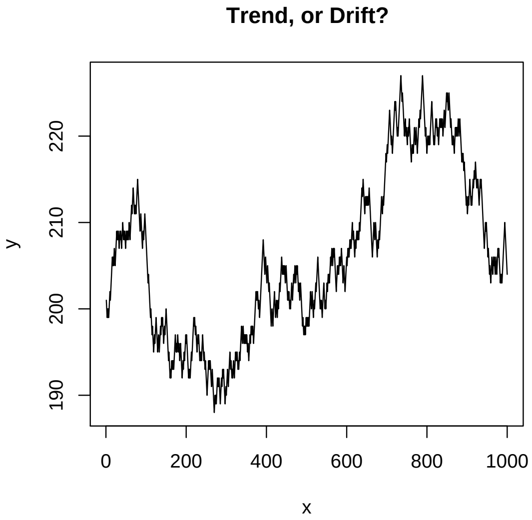
- Not a random sample.
- Not necessarily independent.
- Sequentially dependent.

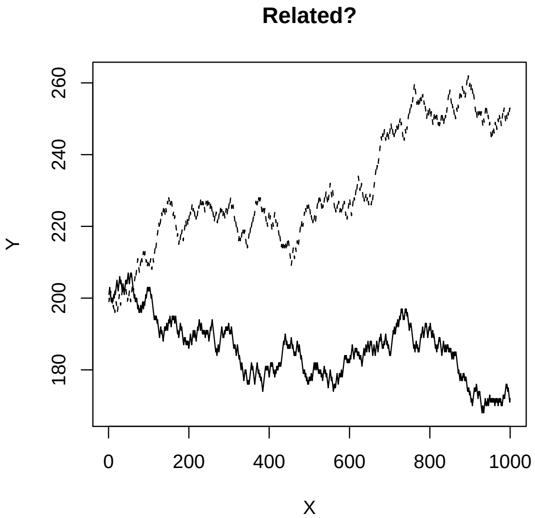
# Trend, or Drift?

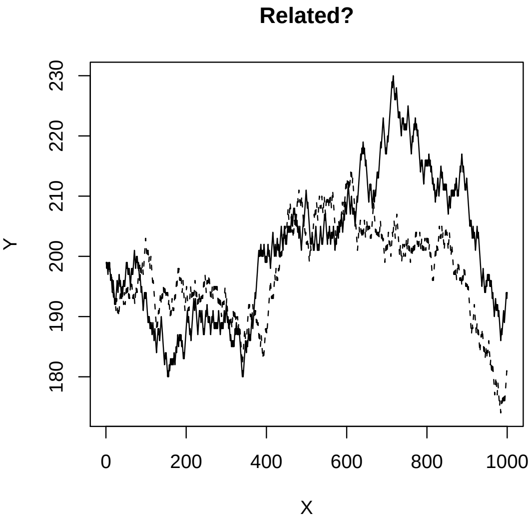




Trend, or Drift?
y
170
180
190
200
0
200
400
600
800
1000
x


Trend, or Drift?
y
190
200
210
220
x
0
200
400
600
800
1000


Related?
Y
260
240
220
200
180
0
200
400
600
800
1000
X


Related?
Y
230
220
210
200
190
180
0
200
400
600
800
1000
X

# Correlations: 50 pairs of independent random walks, $n = 1000$ steps

Need around $|r| = 0.13$ for significance

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| -0.28175 | -0.22242 | -0.32170 | -0.45053 | 0.07866 | 0.59167 | -0.27414 | -0.82570 |
| -0.62175 | 0.43537 | 0.84147 | 0.04103 | -0.17502 | -0.89710 | -0.19116 | -0.53865 |
| -0.50889 | 0.42855 | -0.91074 | 0.90577 | 0.22818 | 0.84834 | -0.52501 | 0.82583 |
| -0.06838 | -0.00234 | 0.16084 | 0.81393 | -0.07063 | -0.09908 | -0.38405 | -0.28510 |
| 0.24850 | 0.12445 | 0.33509 | 0.33586 | 0.41241 | -0.33482 | -0.32021 | -0.73808 |
| 0.14045 | -0.03618 | -0.67757 | 0.81121 | -0.39379 | -0.58832 | -0.26866 | 0.16687 |
| 0.38541 | 0.12433 | | | | | | |

# Random walk

## Sometimes called Drunkard's walk

- Take a step left or right at random.
- Steps could be of variable length.
- Location at time $t$ depends on location at time $t-1$.

$$X_t = X_{t-1} + \epsilon_t$$

$\epsilon_1, \epsilon_2, \ldots$ all independent and identically distributed.

# Autoregressive Time Series

## A generalization of the random walk

| | |
|---|---|
| $X_t = X_{t-1} + \epsilon_t$ | Random walk |
| $X_t = \beta_0 + \beta_1 X_{t-1} + \epsilon_t$ | First order autogrgressive |
| $X_t = \beta_0 + \beta_1 X_{t-1} + \beta_2 X_{t-2} + \epsilon_t$ | Second order autogrgressive |

etc.

# Stationary Time Series

- In a stationary time series, the distribution of $X_t$ is not changing.
- In particular, all the $X_t$ have the same mean and variance.

# Expected value does not change

$$\begin{aligned}
E(X_t) &= E(\beta_0 + \beta_1 X_{t-1} + \epsilon_t) \\
&= \beta_0 + \beta_1 E(X_{t-1}) + 0 \\
\Rightarrow \mu &= \beta_0 + \beta_1 \mu \\
\Rightarrow \beta_0 &= \mu(1 - \beta_1)
\end{aligned}$$

# Variance does not change

$$
\begin{aligned}
Var(X_t) &= Var(\beta_0 + \beta_1 X_{t-1} + \epsilon_t) \\
&= \beta_1^2 Var(X_{t-1}) + Var(\epsilon_t) \\
\Rightarrow \sigma^2 &= \beta_1^2 \sigma^2 + Var(\epsilon_t) \\
\Rightarrow Var(\epsilon_t) &= \sigma^2(1 - \beta_1^2)
\end{aligned}
$$

# Covariance

$$
\begin{aligned}
Cov(X_{t-1}, X_t) &= Cov(X_{t-1}, \beta_0 + \beta_1 X_{t-1} + \epsilon_t) \\
&= \beta_1 Cov(X_{t-1}, X_{t-1}) + Cov(X_{t-1}, \epsilon_t) \\
&= \beta_1 Var(X_{t-1}) + 0 \\
&= \beta_1 \sigma^2
\end{aligned}
$$

So

$$
\begin{aligned}
Corr(X_{t-1}, X_t) &= \frac{\beta_1 \sigma^2}{\sqrt{\sigma^2}\sqrt{\sigma^2}} \\
&= \beta_1
\end{aligned}
$$

# $Corr(X_t, X_{t-1}) = \beta_1$

Where $X_t = \beta_0 + \beta_1 X_{t-1} + \epsilon_t$

- The regression coefficient $\beta_1$ is usually denoted by $\rho$.
- The **First-order Autocorrelation**.
- Continuing the calculations, get $Corr(X_t, X_{t-2}) = \rho^2, \ldots$
- $Corr(X_t, X_{t-m}) = \rho^m$.
- So the covariance matrix looks like this:

$$\sigma^2 \begin{pmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots \\ \rho & 1 & \rho & \rho^2 & \cdots \\ \rho^2 & \rho & 1 & \rho & \cdots \\ \rho^3 & \rho^2 & \rho & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

# Signatures

## Identifying the times series model

- Because $-1 < \rho < 1$, the pattern $\rho, \rho^2, \rho^3, \ldots$ displays a pattern of *exponential decay*: Graph it.
- Other time series structures have known signatures too.
- Higher-order autoregressive.
- Moving average.
- ARMA: Autoregressive Moving Average.
- Seasonal: Blips at seasonal lags.
- Non-stationary.
- Differencing is a big trick.
- ARIMA: Autoregressive Integrated Moving Average.
- Theorem: All the stationary processes can be approximated by autogregressive with enough lags.

# Time series structures for the *error terms* (epsilons) in a regression

- What is the error term $\epsilon$ in a regression?
- Everything that affects $y$ other than the $x$ variables.
- Maybe those omitted variables are sequentially dependent.
- Like the temperature influences pop sales.
- Is it likely? Depends on the logic of the data collection.
- Diagnose by the Durbin-Watson test and time series diagnostics on the residuals.

# Durbin-Watson Test for Autocorrelation

- Usually, autocorrelation is positive.
- $H_0 : \rho = 0$ vs. $H_1 : \rho > 0$

$$D = \frac{\sum_{i=2}^{n}(e_i - e_{i-1})^2}{\sum_{i=1}^{n} e_i^2}$$

- Reject when $D$ is small. How small?
- Critical values and $p$-values are brutally hard to compute.
- Durbin and Watson published tables with upper and lower bounds for the critical values!
- Now SAS can compute the "exact" $p-$values, but it's an option.

# What to do about autocorrelated residuals

- Try adding more explanatory variables, perhaps including time.
- Consider differencing.
- Directly model autocorrelated errors.

# `proc autoreg`

- Regression model with autogregressive errors: covers a lot of important cases.
- Especially in combination with *lagged* explanatory variables.
- Estimate $\beta_j$ and $\rho_k$ all at once by maximum likelihood.
- `proc autoreg` has many capabilities. As usual, we will explore just a few.
- Can you say GARCH?

# Copyright Information