Permutation and Randomization Tests[1]

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Overview

The lady and the tea

From Fisher's *The design of experiments*, first published in 1935

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- The lady tasted them, and judged.
- She knew there were four of each type.

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- The test statistic is the number of correct judgements.
- What is the distribution of the test statistic under the null hypothesis?

Data file

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  Truth Judgement
1   tea      milk
2  milk       tea
3  milk      milk
4  milk      milk
5   tea       tea
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8  milk       tea
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- But there are lots of repeats.

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- So H_0 would be rejected at $\alpha = 0.05$ if she guessed perfectly.

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- The only reason for differences among conditions is the random assignment.

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- Make a histogram.

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- Do a matched t-test, or …

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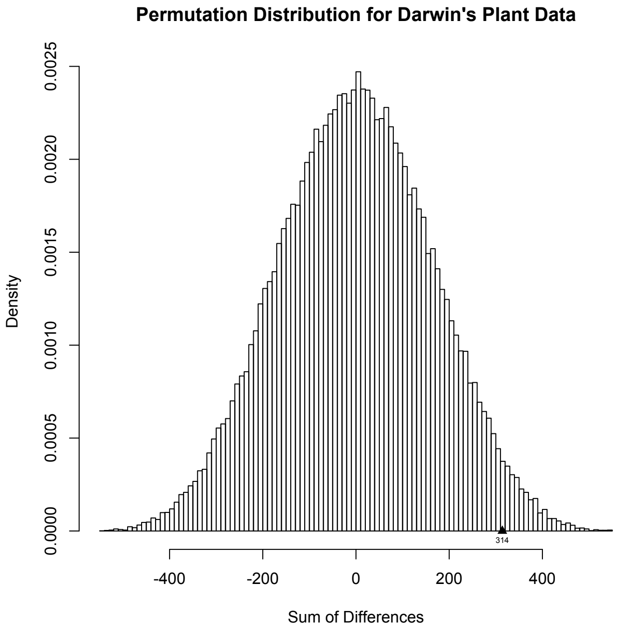
- He used his brain as well as doing a lot of tedious calculation.


Permutation Distribution for Darwin's Plant Data
Density
0.0000
0.0005
0.0010
0.0015
0.0020
0.0025
314
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0
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Sum of Differences

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- Random assignment actually happens. Random sampling often does not.

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- If either explanatory or response variable is multivariate, scramble *vectors* of data.

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- The answer is given by the permutation p-value.

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Choice of test statistic

- We are testing a null hypothesis based on the value of the test statistic.
- The probability of wrongly rejecting H_0 (and making a false discovery) is limited to $\alpha = 0.05$. Good.
- Some test statistics are better than others, depending on *how* H_0 is false: Statistical power.
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- When the assumptions happen to be approximately satisfied, they often are nearly optimal.

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1. Compute some test statistic using the set of original observations.
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Using the p-value from a traditional test

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- One-sided, two-sided does not matter.
- Handy for multiple comparisons (More later).

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See Cox and Reid (2000) *The Theory of the Design of Experiments* for the research literature.

Scab disease data

Illustrating Fisher's claim

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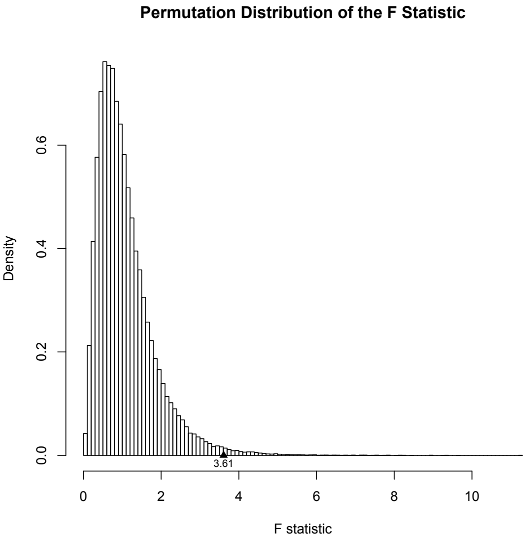

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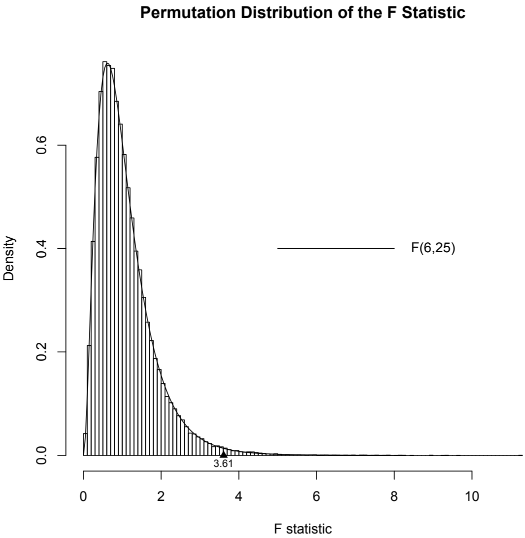
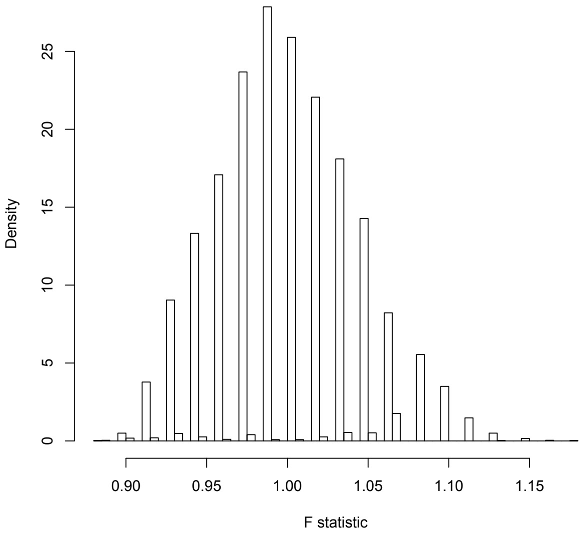
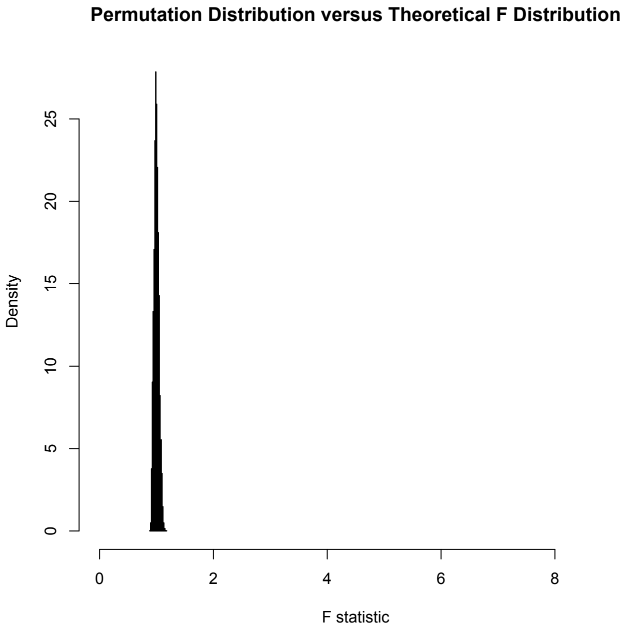


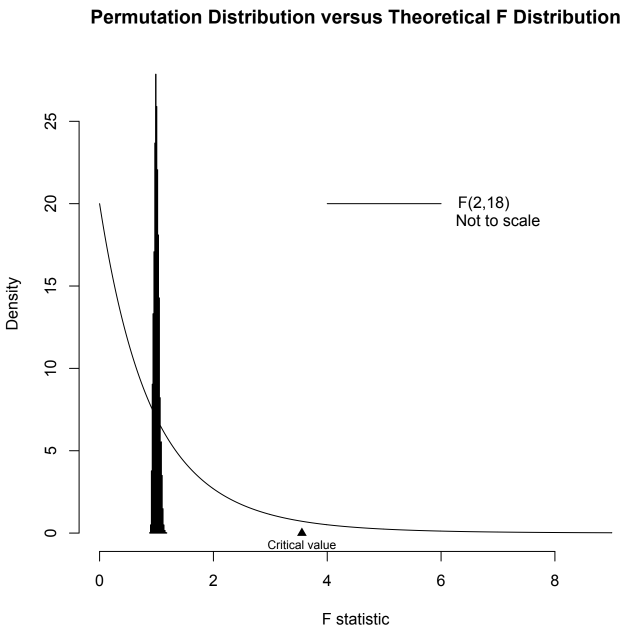
The approximation is not always so good

Group1	Group2	Group3
220	1	4
0	0	0
1	2	0
0	4	3
1	2	1
1	0	4
0	1	1

Permutation Distribution of the F Statistic




Permutation Distribution versus Theoretical F Distribution
Density
0
5
10
15
20
25
0
2
4
6
8
F statistic


Permutation Distribution versus Theoretical F Distribution
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F(2,18)
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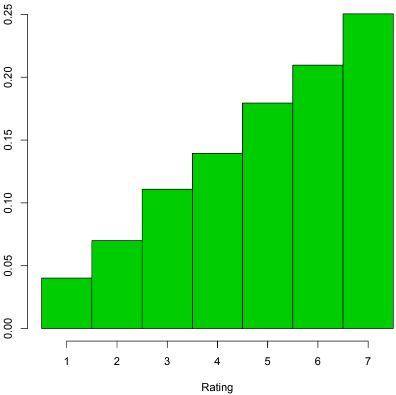
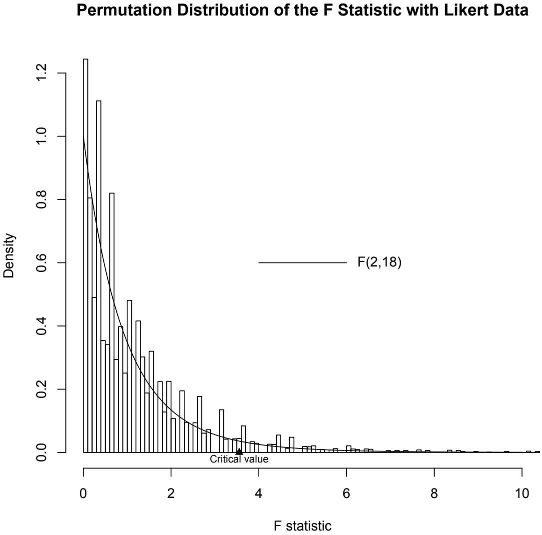


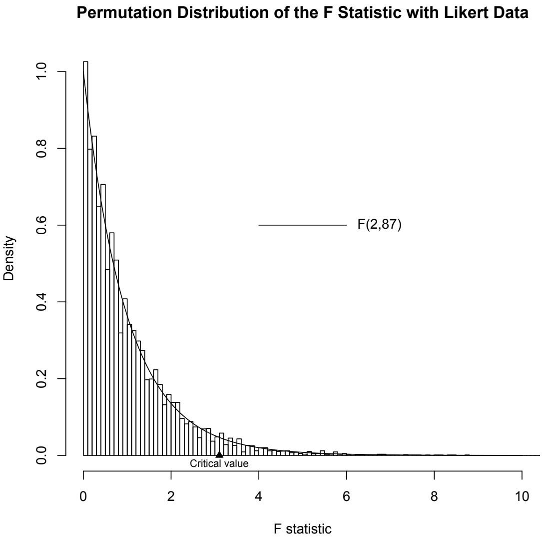
$n = 7$ for each of three treatments

0.0542 of the permutation distribution is above the F critical vlue



$n = 30$ for each of three treatments

0.0472 of the permutation distribution is above the F critical vlue



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- Even with computers, listing all the permutations can be out of the question, and combinatoric simplification may be challenging.

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- For non-binary response variables, one can convert the data to ranks.
- Then, permutation distributions can be figured out in advance.
- All the common non-parametric rank tests are permutation tests carried out on ranks.

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- SAS does this (among other options) in `proc npar1way`.
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- You do need to know what all the tests are, in advance.

Use a standard p-value as the test statistic

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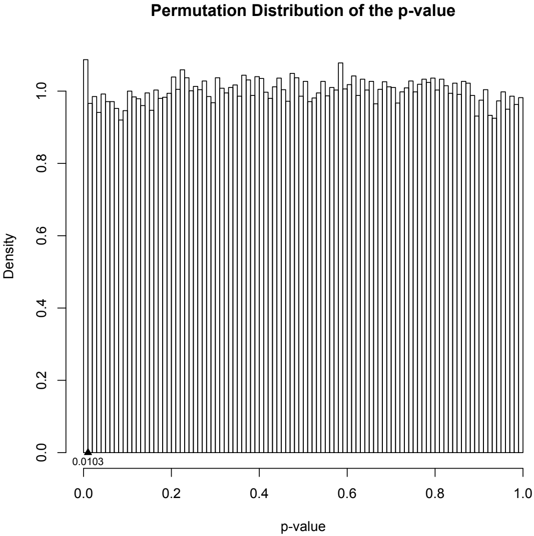

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- Except the p-values are not independent.

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- Give or take discreteness and Monte Carlo sampling error.
- `proc multtest` does this.

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- If the sample size is large enough, the histogram of the sample data is a lot like the histogram of the entire population.
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- Sampling from the sample is called **resampling**.

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- Use it to construct tests and confidence intervals.

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