

# Logistic regression with more than two outcomes: The multinomial logit model

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If there are  $k$  outcomes

Think of  $k-1$  dummy variables for  
the response variable

# Model for three categories

$$\ln \left( \frac{\pi_1}{\pi_3} \right) = \beta_{0,1} + \beta_{1,1}x_1 + \dots + \beta_{p-1,1}x_{p-1}$$

$$\ln \left( \frac{\pi_2}{\pi_3} \right) = \beta_{0,2} + \beta_{1,2}x_1 + \dots + \beta_{p-1,2}x_{p-1}$$

Need  $k-1$  **generalized logits** to represent a response variable with  $k$  categories

# Meaning of the regression coefficients

$$\ln \left( \frac{\pi_1}{\pi_3} \right) = \beta_{0,1} + \beta_{1,1}x_1 + \dots + \beta_{p-1,1}x_{p-1}$$

$$\ln \left( \frac{\pi_2}{\pi_3} \right) = \beta_{0,2} + \beta_{1,2}x_1 + \dots + \beta_{p-1,2}x_{p-1}$$

A positive regression coefficient for logit  $j$  means that higher values of the explanatory variable are associated with greater chances of response category  $j$ , as opposed to the reference category.

# Solve for the probabilities

$$\ln \left( \frac{\pi_1}{\pi_3} \right) = L_1 \quad \text{so} \quad \frac{\pi_1}{\pi_3} = e^{L_1}$$

$$\ln \left( \frac{\pi_2}{\pi_3} \right) = L_2 \quad \frac{\pi_2}{\pi_3} = e^{L_2}$$

$$\pi_1 = \pi_3 e^{L_1}$$

So

$$\pi_2 = \pi_3 e^{L_2}$$

# Three linear equations in 3 unknowns

$$\pi_1 = \pi_3 e^{L_1}$$

$$\pi_2 = \pi_3 e^{L_2}$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

# Solution

$$\pi_1 = \frac{e^{L_1}}{1 + e^{L_1} + e^{L_2}}$$

$$\pi_2 = \frac{e^{L_2}}{1 + e^{L_1} + e^{L_2}}$$

$$\pi_3 = \frac{1}{1 + e^{L_1} + e^{L_2}}$$

In general, solve  $k$  equations  
in  $k$  unknowns

$$\begin{aligned}\pi_1 &= \pi_k e^{L_1} \\ &\vdots \\ \pi_{k-1} &= \pi_k e^{L_{k-1}} \\ \pi_1 + \cdots + \pi_k &= 1\end{aligned}$$



# General Solution

$$\begin{aligned}\pi_1 &= \frac{e^{L_1}}{1 + \sum_{j=1}^{k-1} e^{L_j}} \\ \pi_2 &= \frac{e^{L_2}}{1 + \sum_{j=1}^{k-1} e^{L_j}} \\ &\vdots \\ \pi_{k-1} &= \frac{e^{L_{k-1}}}{1 + \sum_{j=1}^{k-1} e^{L_j}} \\ \pi_k &= \frac{1}{1 + \sum_{j=1}^{k-1} e^{L_j}}\end{aligned}$$

# Using the solution, one can

- Calculate the probability of obtaining the observed data as a function of the regression coefficients: Get maximum likelihood estimates ( $b$  values)
- From maximum likelihood estimates, get large-sample tests and confidence intervals
- Using  $b$  values in  $L_j$ , estimate probabilities of category membership for any set of  $x$  values.

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