Logistic Regression

For a binary response variable:
1=Yes, 0=No

Binary outcomes are common and important

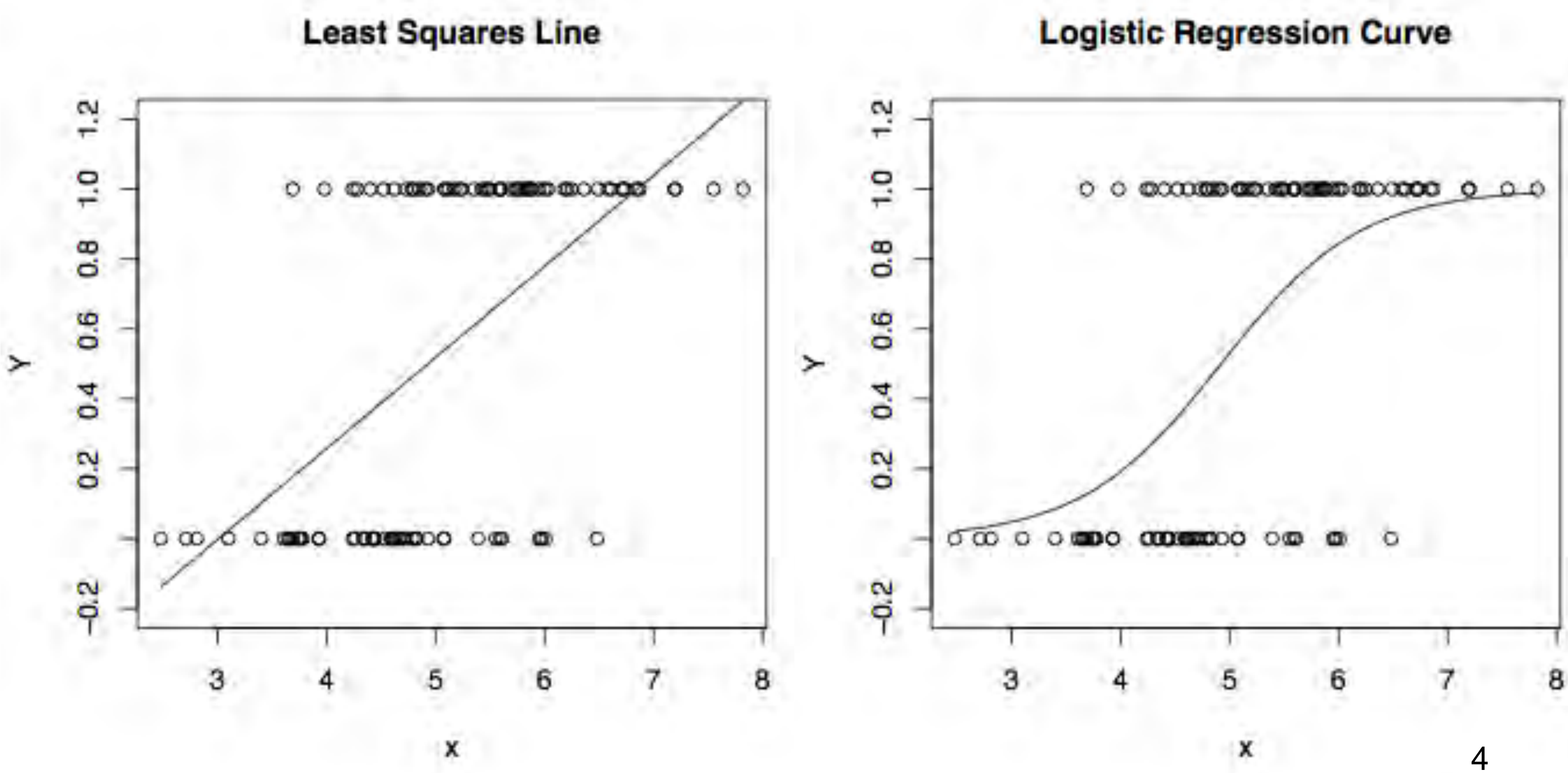
- The patient survives the operation, or does not.
- The accused is convicted, or is not.
- The customer makes a purchase, or does not.
- The marriage lasts at least five years, or does not.
- The student graduates, or does not.

For a binary variable

- The population mean E[Y] is the probability that Y=1
- Make the mean depend on a set of explanatory variables

- Consider one explanatory variable. Think of a scatterplot

Least Squares vs. Logistic Regression



The logistic regression curve arises from an indirect representation of the probability of Y=1 for a given set of x values.

Representing the probability of an event by π

$$\text{Odds} = \frac{\pi}{1-\pi}$$

$$\text{Odds} = \frac{\pi}{1 - \pi}$$

- If P(Y=1)=1/2, odds = .5/(1-.5) = 1 (to 1)
- If P(Y=1)=2/3, odds = 2 (to 1)
- If P(Y=1)=3/5, odds = (3/5)/(2/5) = 1.5 (to 1)
- If P(Y=1)=1/5, odds = .25 (to 1)

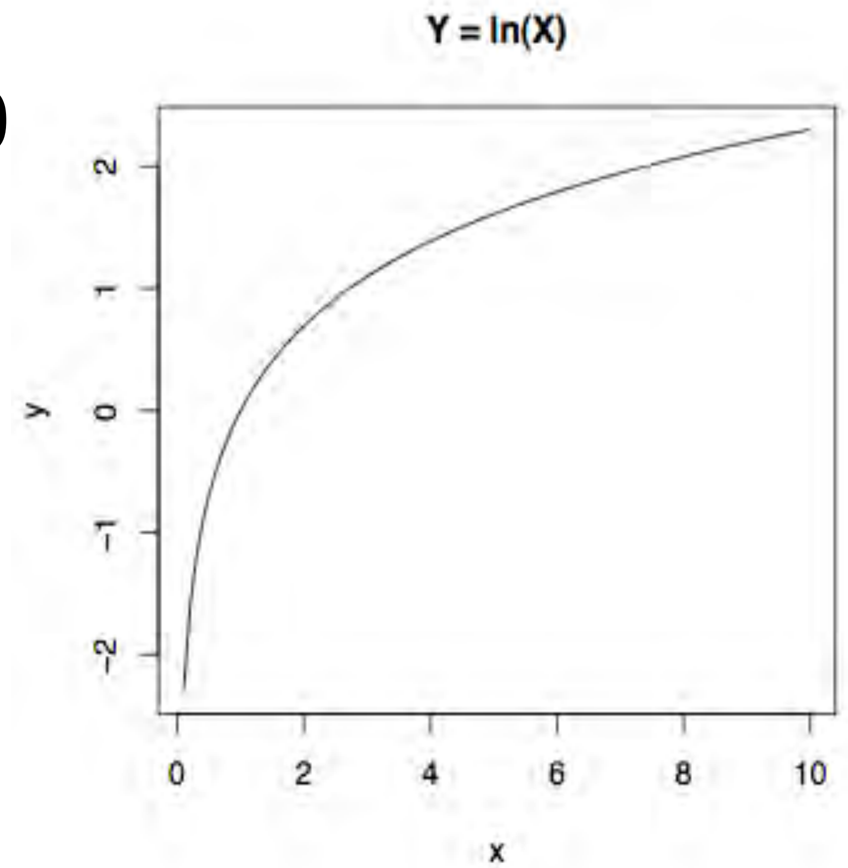
The higher the probability, the greater the odds

$$\text{Odds} = \frac{\pi}{1-\pi}$$

$$0 \leq \text{Odds} < \infty$$

Linear model for the **log** odds

- Natural log, not base 10
- Symbolized `ln`



- The higher the probability, the higher the log odds.

Linear regression model for the log odds of the event Y=1

$$\ln\left(\frac{P(Y=1|\mathbf{X}=\mathbf{x})}{P(Y=0|\mathbf{X}=\mathbf{x})}\right) = \beta_0 + \beta_1 x_1 + \ldots + \beta_{p-1} x_{p-1}$$

Probability zero or one is excluded

$$\ln\left(\frac{P(Y=1|\mathbf{X}=\mathbf{x})}{P(Y=0|\mathbf{X}=\mathbf{x})}\right) = \beta_0 + \beta_1 x_1 + \ldots + \beta_{p-1} x_{p-1}$$

- Log is only defined for positive numbers.
- So any model for the log odds, including logistic regression, will not work for events of probability exactly zero or exactly one.
- Why not one?

Equivalent Statements

$$\ln\left(\frac{P(Y=1|\mathbf{X}=\mathbf{x})}{P(Y=0|\mathbf{X}=\mathbf{x})}\right) = \beta_0 + \beta_1 x_1 + \ldots + \beta_{p-1} x_{p-1}$$

$$\begin{aligned}\frac{P(Y=1|\mathbf{X}=\mathbf{x})}{P(Y=0|\mathbf{X}=\mathbf{x})} &= e^{\beta_0+\beta_1 x_1+\ldots+\beta_{p-1}x_{p-1}} \\ &= e^{\beta_0} e^{\beta_1 x_1} \cdots e^{\beta_{p-1}x_{p-1}}\end{aligned}$$

$$P(Y=1|x_1,\ldots,x_{p-1}) = \frac{e^{\beta_0+\beta_1 x_1+\ldots+\beta_{p-1}x_{p-1}}}{1+e^{\beta_0+\beta_1 x_1+\ldots+\beta_{p-1}x_{p-1}}}$$

In terms of log odds, logistic regression is like regular regression

$$\ln\left(\frac{P(Y=1|\mathbf{X}=\mathbf{x})}{P(Y=0|\mathbf{X}=\mathbf{x})}\right) = \beta_0 + \beta_1 x_1 + \ldots + \beta_{p-1} x_{p-1}$$

In terms of plain odds,

- Logistic regression coefficients are related to *odds ratios.*
- For example, "Among 50 year old men, the odds of being dead before age 60 are three times as great for smokers."

$$\frac{\text{Odds of death given smoker}}{\text{Odds of death given nonsmoker}} = 3$$

Logistic regression

- X=1 means smoker, X=0 means non-smoker
- Y=1 means dead, Y=0 means alive
- Log odds of death = $\beta_0 + \beta_1 x$
- Odds of death = $e^{\beta_0} e^{\beta_1 x}$

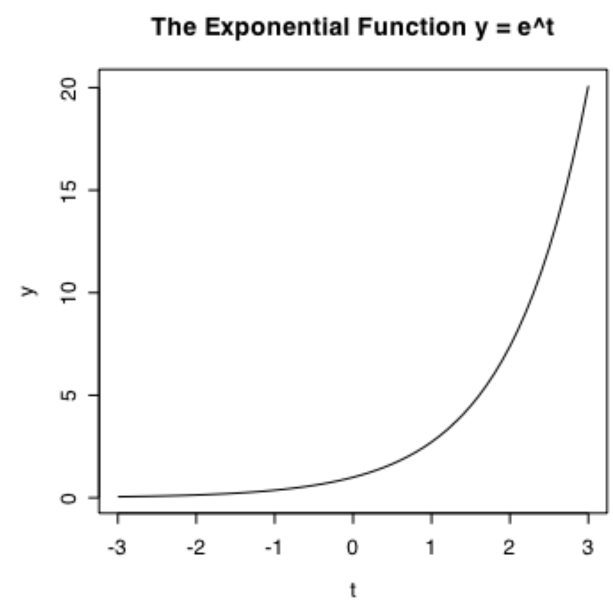
$$\text{Odds of Death} = e^{\beta_0} e^{\beta_1 x}$$

Group	x	**Odds of Death**
Smokers	1	$e^{\beta_0} e^{\beta_1}$
Non-smokers	0	e^{β_0}

$$\frac{\text{Odds of death given smoker}}{\text{Odds of death given nonsmoker}} = \frac{e^{\beta_0} e^{\beta_1}}{e^{\beta_0}} = e^{\beta_1}$$

Exponential function $f(t) = e^t$

- Always positive
- $e^0=1$, so when $\beta_1 = 0$, the odds ratio $e^{\beta 1}$ equals one (50-50).
- $f(t) = e^t$ is increasing



Another example

$$\text{Log Survival Odds} = \beta_0 + \beta_1 d_1 + \beta_2 d_2 + \beta_3 x$$

Treatment	d_1	d_2	**Odds of Survival** $= e^{\beta_0} e^{\beta_1 d_1} e^{\beta_2 d_2} e^{\beta_3 x}$
Chemotherapy	1	0	$e^{\beta_0} e^{\beta_1} e^{\beta_3 x}$
Radiation	0	1	$e^{\beta_0} e^{\beta_2} e^{\beta_3 x}$
Both	0	0	$e^{\beta_0} e^{\beta_3 x}$

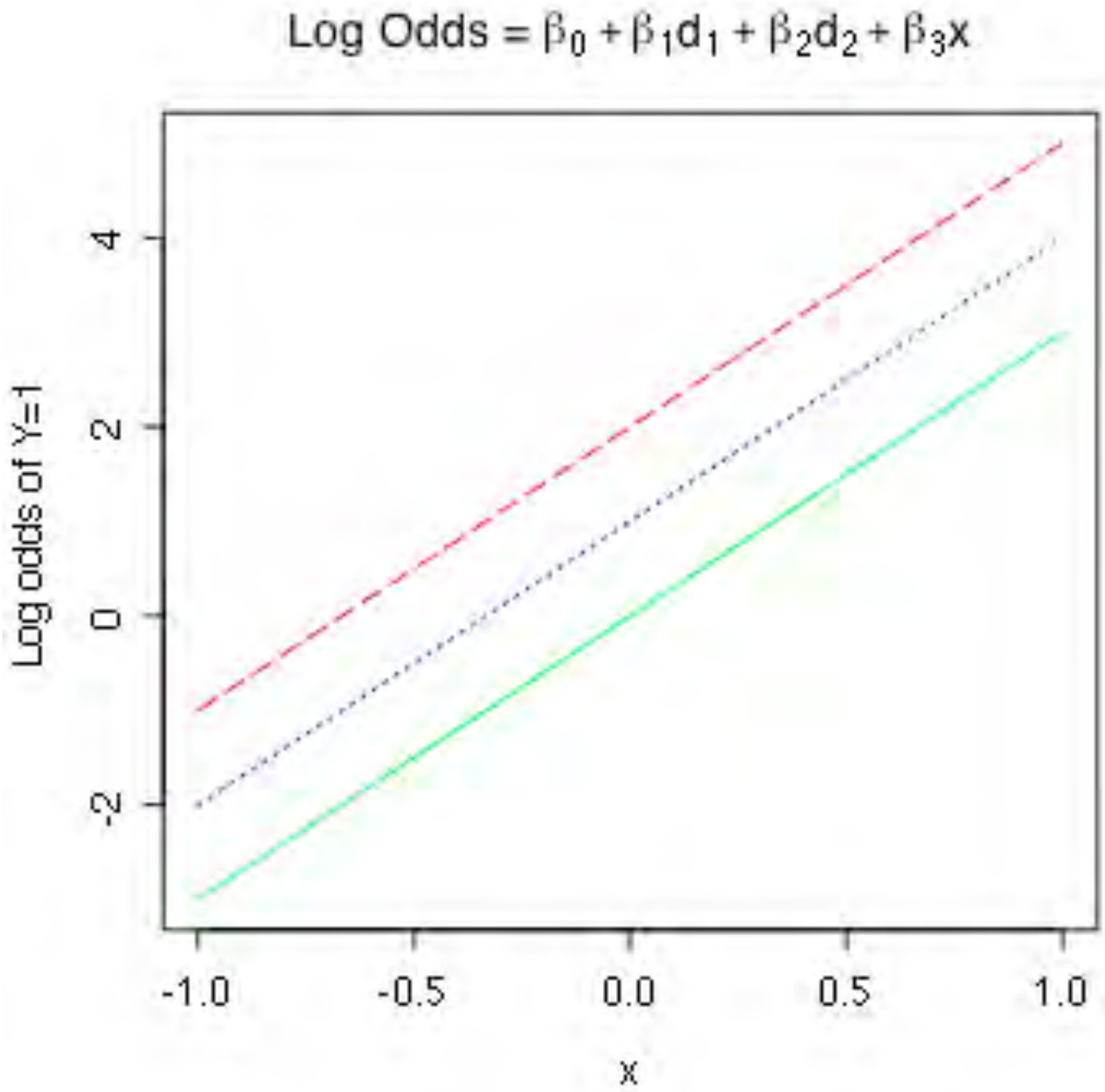
For any given disease severity x,

$$\frac{\text{Survival odds with Chemo}}{\text{Survival odds with Both}} = \frac{e^{\beta_0} e^{\beta_1} e^{\beta_3 x}}{e^{\beta_0} e^{\beta_3 x}} = e^{\beta_1}$$

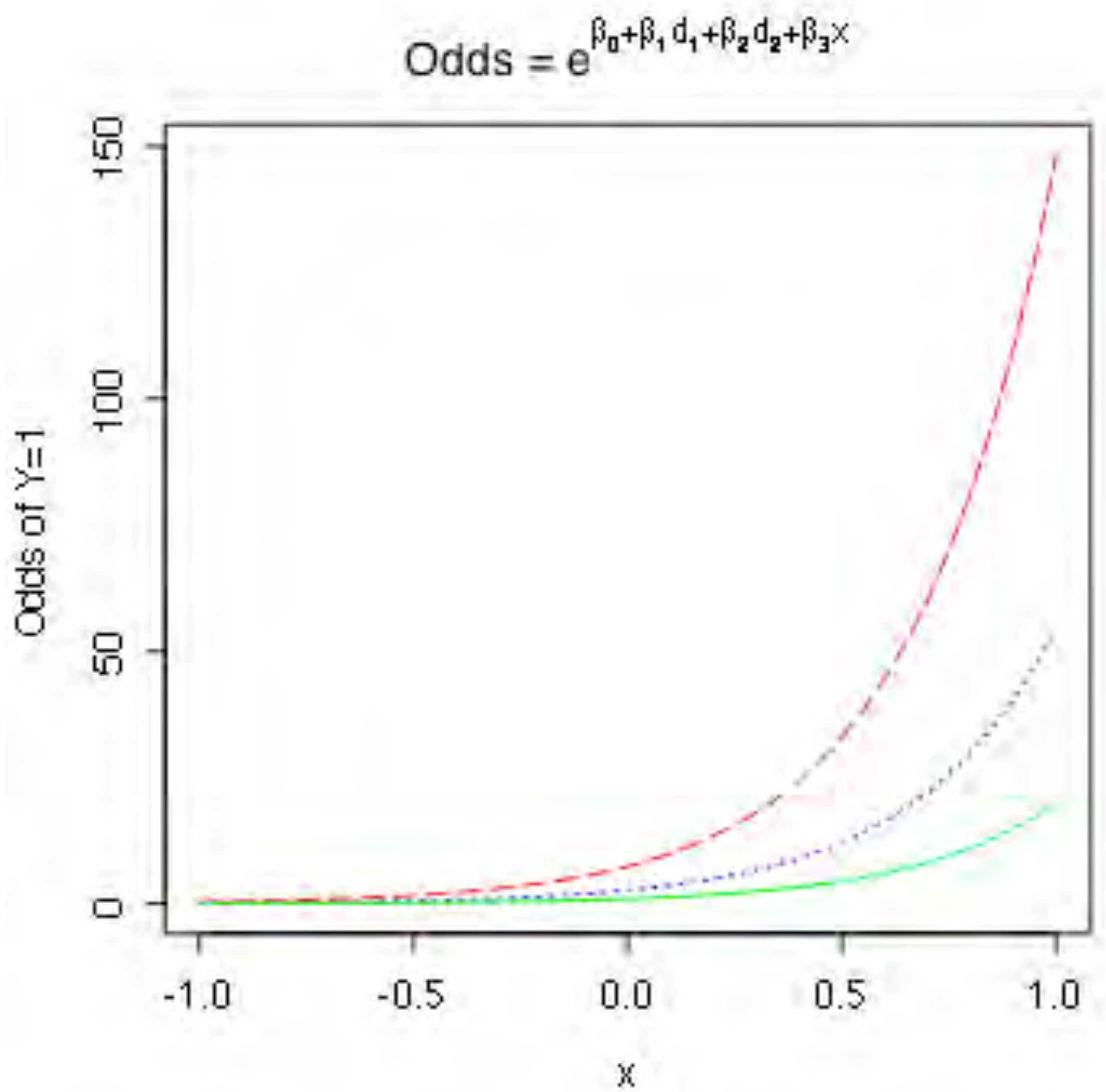
In general,

- When x_k is increased by one unit and all other explanatory variables are held constant, the odds of Y=1 are multiplied by e^{β_k}
- That is, e^{β_k} is an **odds ratio** --- the ratio of the odds of Y=1 when x_k is increased by one unit, to the odds of Y=1 when everything is left alone.
- As in ordinary regression, we speak of "controlling" for the other variables.

Equal slopes in the log odds scale

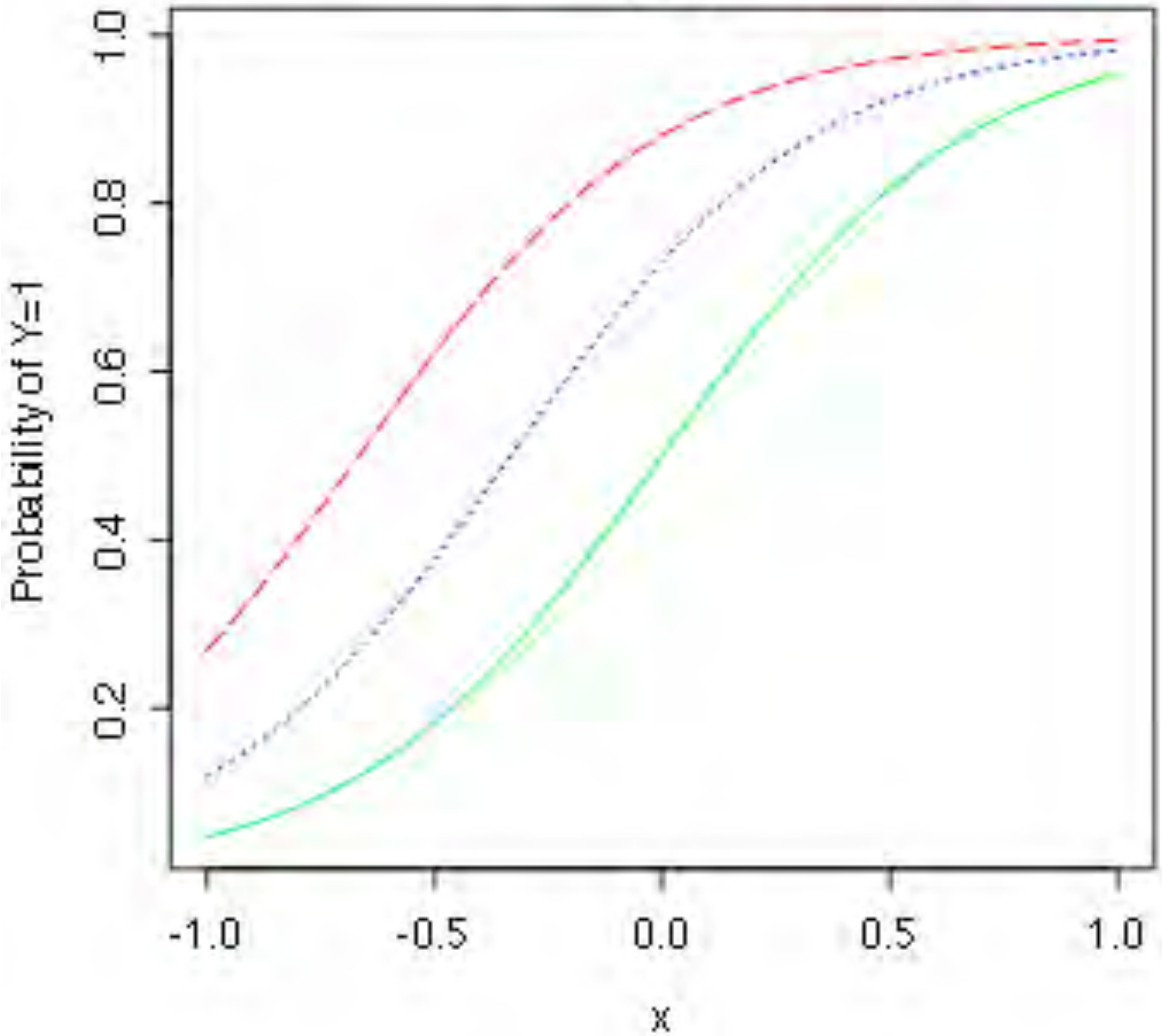


Equal slopes in the log odds scale means proportional odds



Proportional Odds in Terms of Probability

$$\text{Probability} = \frac{e^{\beta_0+\beta_1 d_1+\beta_2 d_2+\beta_3 x}}{1+e^{\beta_0+\beta_1 d_1+\beta_2 d_2+\beta_3 x}}$$



Interactions

- With equal slopes in the log odds scale, *differences* in odds and *differences* in probabilities do depend on x.
- Regression coefficients for product terms still mean something.
- If zero, they mean that the *odds ratio* does not depend on the value(s) of the covariate(s).
- Odds ratio has odds of Y=1 for the reference category in the denominator.
- Most of our models will not have product terms.

The conditional probability of Y=1

$$P(Y=1|x_1,\ldots,x_{p-1}) = \frac{e^{\beta_0+\beta_1 x_1+\ldots+\beta_{p-1}x_{p-1}}}{1+e^{\beta_0+\beta_1 x_1+\ldots+\beta_{p-1}x_{p-1}}}$$

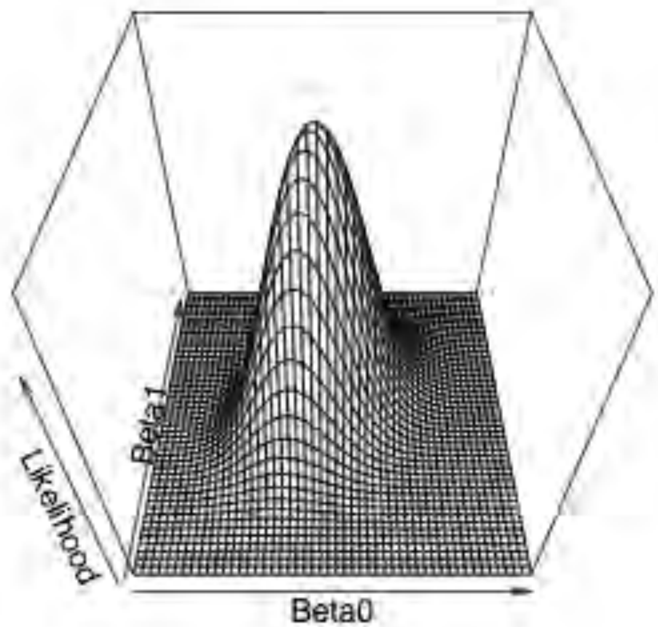
This formula can be used to calculate an estimated P(Y=1)
Just replace betas by their estimates (b)

It can also be used to calculate the probability of getting
The sample data values we actually did observe.

Maximum likelihood estimation

- Likelihood = Probability of getting the data values we did observe
- Viewed as a function of the parameters (betas), it's called the "likelihood function."
- Those parameter values for which the likelihood function is greatest are called the *maximum likelihood estimates.*
- Thank you again, Mr. Fisher.

Likelihood Function for Simple Logistic Regression



Maximum likelihood estimates

- Must be found numerically.
- For the record, using “iteratively re-weighted least squares.”
- Lead to nice large-sample chi-square tests.
- Most common are likelihood ratio tests and Wald tests.
- We will mostly use Wald tests.

Likelihood Ratio Tests

- Likelihood at MLE is the maximum probability of obtaining the observed data.
- Higher probability means better model fit, but they are all very small.
- -2 log likelihood measures lack of fit.
- Restricted (reduced) model always fits worse than unrestricted (full).
- $G2 = -2LL_R \; - \; -2LL_F$
- df is number of = signs in H_0.

Wald tests

- Based directly on approximate large-sample normality of the MLE.
- Thank you, Mr. Wald.
- Formula looks like the numerator of the general linear F-test statistic.
- Wald and LR tests are asymptotically equivalent under H_0.
- Meaning that if H_0 is true, the difference between the test statistics goes to zero in probability as $n \rightarrow \infty$.
- If H_0 is false, they both go to ∞ but need not be close.
- LR tests perform better for smaller samples, and have other advantages.
- We will mostly use Wald tests because SAS makes them more convenient.

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