Name Jerry

Student Number

STA 431 Quiz 6

In homework, you examined the reliability of a long test with equivalent items, meaning that the variances of the error terms were all the same. The following model is more realistic. Independently for $i = 1, \ldots, n$, let

$$W_i = \nu_i + \lambda_i X + e_i,$$

where $E(X) = \mu$, $Var(X) = \phi$, $E(e_i) = 0$, and $Var(e_i) = \omega_i$. The e_i are independent of one another, and they are independent of X. See the footnote[1] for the intuition behind the model, if you are interested.

The parameters ν_i, λ_i and ω_i are fixed constants, and we need to make them reasonably well-behaved. Assume that as $n \to \infty$,

$$\overline{\lambda}_n = \frac{1}{n}\sum_{i=1}^n \lambda_i \to \lambda \neq 0 \quad \text{and} \quad \overline{\omega}_n = \frac{1}{n}\sum_{i=1}^n \omega_i \to \omega$$

These are ordinary limits.

1. (1 point) Score on the entire test will be the average $\overline{W}_n = \frac{1}{n}\sum_{i=1}^n W_i$. To make Question 3 easier, simplify $\overline{W}_n$. Use the formula for W_i above.

$$\overline{W}_n = \frac{1}{n}\sum_{i=1}^n W_i = \frac{1}{n}\sum_{i=1}^n (\nu_i + \lambda_i X + e_i)$$

$$= \frac{1}{n}\sum_{i=1}^n \nu_i + X\frac{1}{n}\sum_{i=1}^n \lambda_i + \frac{1}{n}\sum_{i=1}^n e_i$$

$$= \overline{\nu}_n + \overline{\lambda}_n X + \overline{e}_n$$

2. (1 point) Calculate the variance of $\overline{e}_n = \frac{1}{n}\sum_{i=1}^n e_i$.

$$Var(\overline{e}_n) = Var\left(\frac{1}{n}\sum_{i=1}^n e_i\right) = \frac{1}{n^2}\sum_{i=1}^n Var(e_i)$$

$$= \frac{1}{n^2}\sum_{i=1}^n \omega_i = \frac{1}{n}\overline{\omega}_n$$

This last step can be done in Question 3 instead of here

[1]In this model, X is the true latent quantity being measured by a test or exam. The test has n questions, or "items." The observable variables $W_1, \ldots, W_n$ are the number of points the student gets on each item. These numbers are not just the truth plus a piece of random noise. Each item has its own slope and intercept, and also the error terms have different variances. This is reasonable. Some test questions are worth more marks (that's ν_i), and some are better than others.

3. (6 points) Calculate ρ_n^2, the reliability of $\overline{W}_n$. Show your work and *simplify*. Circle your final answer.

$$\rho_n^2 = \left(Corr(X, \overline{W}_n)\right)^2 = \left(\frac{Cov(X, \overline{W}_n)}{\sqrt{Var(X)}\sqrt{Var(\overline{W}_n)}}\right)^2$$

$$= \frac{\left(Cov(X, \overline{\lambda}_n X + \overline{\epsilon}_n)\right)^2}{\phi \, Var(\overline{\lambda}_n X + \overline{\epsilon}_n)} = \frac{\left(\overline{\lambda}_n \phi + 0\right)^2}{\phi\left(\overline{\lambda}_n^2 \phi + \frac{1}{n}\overline{\omega}_n\right)}$$

$$= \frac{\overline{\lambda}_n^2 \phi^2}{\phi\left(\overline{\lambda}_n^2 \phi + \frac{1}{n}\overline{\omega}_n\right)} = \frac{\overline{\lambda}_n^2 \phi}{\overline{\lambda}_n^2 \phi + \frac{1}{n}\overline{\omega}_n}$$

4. (2 points) Again denoting the reliability of $\overline{W}_n$ by ρ_n^2, calculate $\lim_{n\rightarrow\infty} \rho_n^2$. Show some of the steps.

$$\lim_{n\to\infty} \rho_n^2 = \lim_{n\to\infty} \frac{\overline{\lambda}_n^2 \phi}{\overline{\lambda}_n^2 \phi + \frac{1}{n}\overline{\omega}_n}$$

$$= \frac{\phi \lim_{n\to\infty} \overline{\lambda}_n^2}{\phi \lim_{n\to\infty} \overline{\lambda}_n^2 + \lim_{n\to\infty}\left(\frac{1}{n}\right) \lim_{n\to\infty} \overline{\omega}_n} = \frac{\lambda^2 \phi}{\lambda^2 \phi + 0 \cdot \omega}$$

$$= \frac{\lambda^2 \phi}{\lambda^2 \phi} = 1$$