Parameter Identifiability for the Latent Model[1]

STA431 Spring 2023

[1] See last slide for copyright information.

The two-stage model: $cov(\mathbf{d}_i) = \boldsymbol{\Sigma}$

All variables are centered

$$
\begin{aligned}
\mathbf{y}_i &= \boldsymbol{\beta}\mathbf{y}_i + \boldsymbol{\Gamma}\mathbf{x}_i + \boldsymbol{\epsilon}_i \\
\mathbf{F}_i &= \left(\begin{array}{c} \mathbf{x}_i \\ \mathbf{y}_i \end{array} \right) \\
\mathbf{d}_i &= \boldsymbol{\Lambda}\mathbf{F}_i + \mathbf{e}_i
\end{aligned}
$$

- $\mathbf{x}_i$ is $p \times 1$, $\mathbf{y}_i$ is $q \times 1$, $\mathbf{d}_i$ is $k \times 1$.
- $cov(\mathbf{x}_i) = \boldsymbol{\Phi}_x$, $cov(\boldsymbol{\epsilon}_i) = \boldsymbol{\Psi}$
- $cov(\mathbf{F}_i) = cov\left(\begin{array}{c} \mathbf{x}_i \\ \mathbf{y}_i \end{array} \right) = \boldsymbol{\Phi} = \left(\begin{array}{cc} \boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12} \\ \boldsymbol{\Phi}_{12}^\top & \boldsymbol{\Phi}_{22} \end{array} \right)$
- $cov(\mathbf{e}_i) = \boldsymbol{\Omega}$

Identify parameter matrices in two steps

It does not really matter which one you do first.

- $\mathbf{y}_i = \boldsymbol{\beta}\mathbf{y}_i + \boldsymbol{\Gamma}\mathbf{x}_i + \boldsymbol{\epsilon}_i$

  $cov(\mathbf{x}_i) = \boldsymbol{\Phi}_x$, $cov(\boldsymbol{\epsilon}_i) = \boldsymbol{\Psi}$
- $\mathbf{d}_i = \boldsymbol{\Lambda}\mathbf{F}_i + \mathbf{e}_i$

  $cov(\mathbf{F}_i) = \boldsymbol{\Phi}$, $cov(\boldsymbol{e}_i) = \boldsymbol{\Omega}$

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1. *Latent model*: Show $\boldsymbol{\beta}$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Phi}_x$ and $\boldsymbol{\Psi}$ can be recovered from $\boldsymbol{\Phi} = cov\left(\begin{array}{c} \mathbf{x}_i \\ \mathbf{y}_i \end{array}\right)$.
2. *Measurement model*: Show $\boldsymbol{\Phi}$, $\boldsymbol{\Lambda}$ and $\boldsymbol{\Omega}$ can be recovered from $\boldsymbol{\Sigma} = cov(\mathbf{d}_i)$.

This means all the parameters can be recovered from $\boldsymbol{\Sigma}$.

Latent Model Rules

- $\mathbf{y}_i = \boldsymbol{\beta}\mathbf{y}_i + \boldsymbol{\Gamma}\mathbf{x}_i + \boldsymbol{\epsilon}_i$
- Here, identifiability means that the parameters $\boldsymbol{\beta}$, $\boldsymbol{\Gamma}$, $\boldsymbol{\Phi}_x$ and $\boldsymbol{\Psi}$ are functions of $cov(\mathbf{F}_i) = \boldsymbol{\Phi}$.

Regression Rule

Someimes called the Null Beta Rule

Suppose

- No endogenous variables influence other endogenous variables.

- $\mathbf{y}_i = \boldsymbol{\Gamma}\mathbf{x}_i + \boldsymbol{\epsilon}_i$
- Of course $cov(\mathbf{x}_i, \boldsymbol{\epsilon}_i) = \mathbf{0}$, always.
- $\boldsymbol{\Psi} = cov(\boldsymbol{\epsilon}_i)$ need not be diagonal.

Then $\boldsymbol{\Gamma}$ and $\boldsymbol{\Psi}$ are identifiable.

With no restriction, the parameters are *just identifiable.* The model is *saturated.*

Acyclic Rule

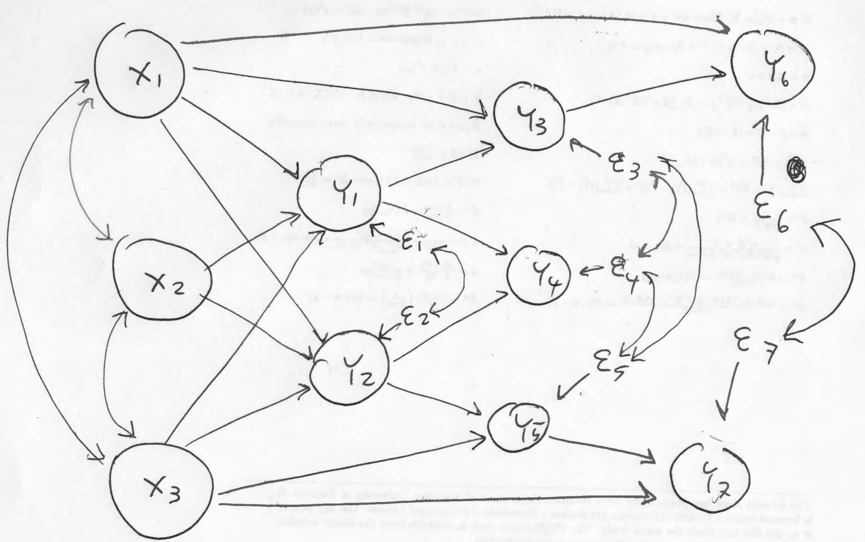
Acyclic models are sometimes called "recursive."

Parameters of the Latent Variable Model are identifiable if the model is acyclic (no feedback loops through straight arrows) and the following conditions hold.

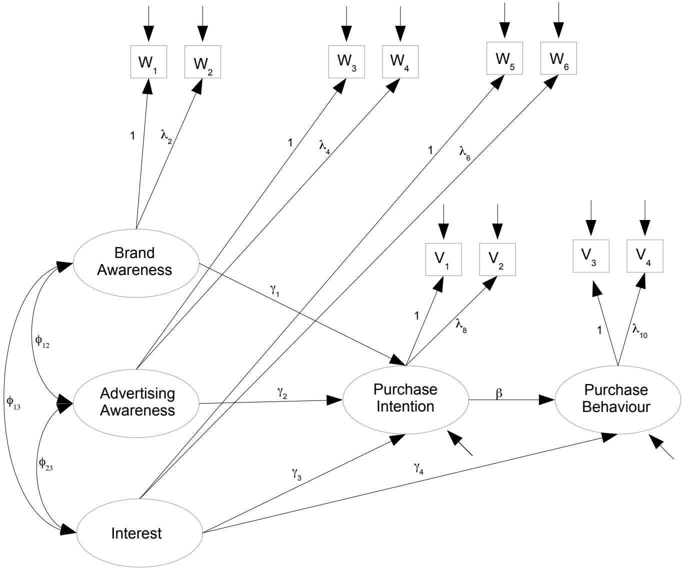
- Organize the variables that are not error terms into sets. Set 0 consists of all the exogenous variables.
- For $j = 1, \ldots, m$, each endogenous variable in set j is influenced by at least one variable in set $j-1$, and also possibly by variables in earlier sets.
- Error terms may be correlated within sets, but not between sets.

Proof: Repeated application of the Regression Rule.

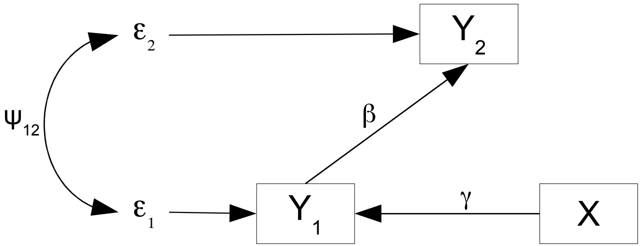
An Acyclic model



Brand awareness model



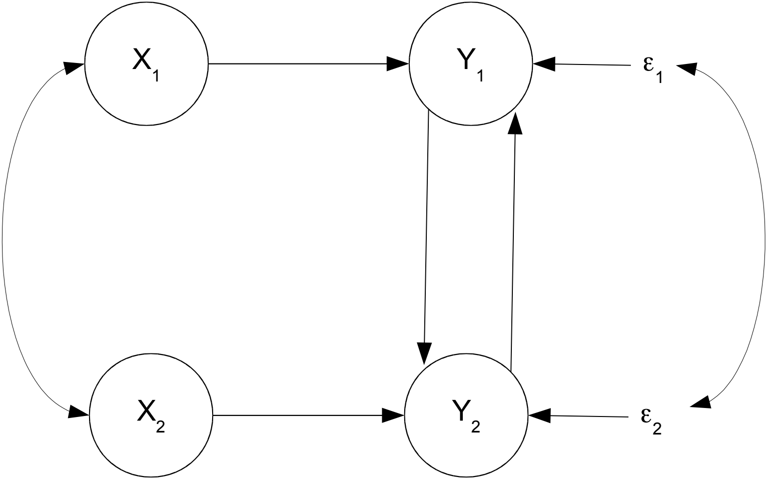
Acyclic Rule Does Not Apply Here



Shows that the acyclic rule is sufficient but not necessary.

Parameters of this model are just identifiable

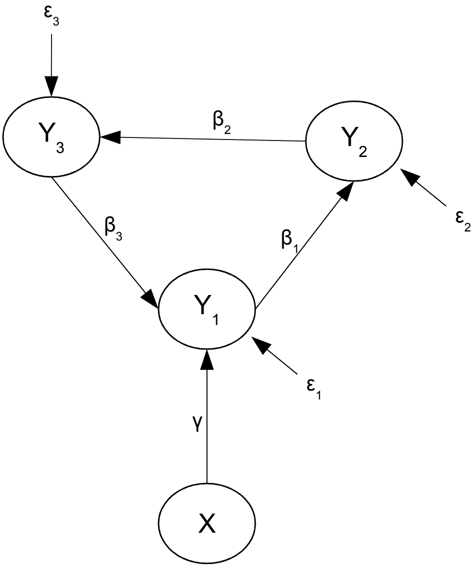
Example from Ch. 5 of Duncan's *Introduction to Structural Equation Models*



Again, the acyclic rule is sufficient but not necessary.

The Pinwheel Model

Parameters are identifiable



Covariance matrix for the pinwheel model

$$
\begin{pmatrix}
\phi & -\frac{\gamma\phi}{\beta_1\beta_2\beta_3-1} & -\frac{\beta_2\gamma\phi}{\beta_1\beta_2\beta_3-1} & -\frac{\beta_2\beta_3\gamma\phi}{\beta_1\beta_2\beta_3-1} \\
 & \frac{\beta_1^2\beta_3^2\psi_2+\gamma^2\phi+\beta_1^2\psi_3+\psi_1}{(\beta_1\beta_2\beta_3-1)^2} & \frac{\beta_2\gamma^2\phi+\beta_1^2\beta_2\psi_3+\beta_1\beta_3\psi_2+\beta_2\psi_1}{(\beta_1\beta_2\beta_3-1)^2} & \frac{\beta_2\beta_3\gamma^2\phi+\beta_1\beta_3^2\psi_2+\beta_2\beta_3\psi_1+\beta_1\psi_3}{(\beta_1\beta_2\beta_3-1)^2} \\
 & & \frac{\beta_2^2\gamma^2\phi+\beta_1^2\beta_2^2\psi_3+\beta_2^2\psi_1+\psi_2}{(\beta_1\beta_2\beta_3-1)^2} & \frac{\beta_2^2\beta_3\gamma^2\phi+\beta_2^2\beta_3\psi_1+\beta_1\beta_2\psi_3+\beta_3\psi_2}{(\beta_1\beta_2\beta_3-1)^2} \\
 & & & \frac{\beta_2^2\beta_3^2\gamma^2\phi+\beta_2^2\beta_3^2\psi_1+\beta_3^2\psi_2+\psi_3}{(\beta_1\beta_2\beta_3-1)^2}
\end{pmatrix}
$$

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