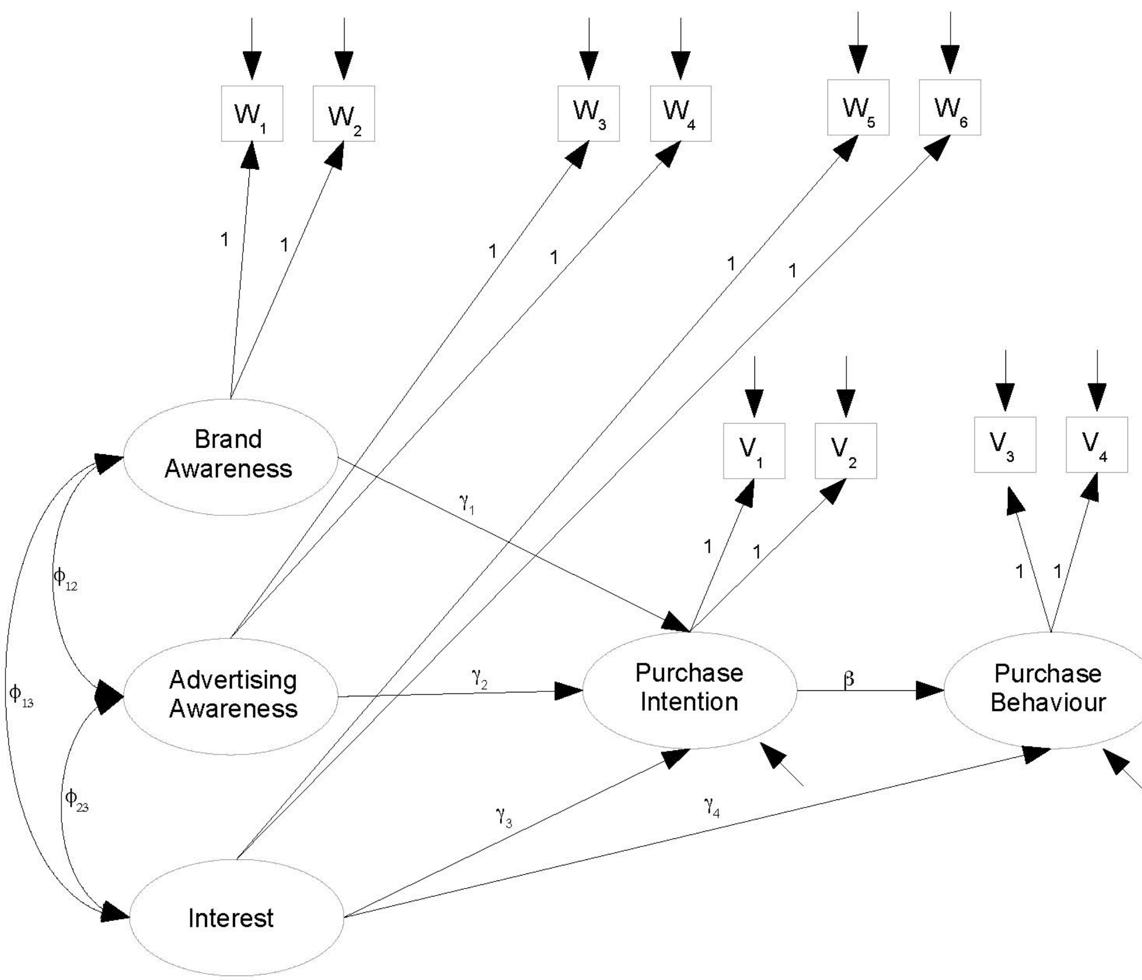
Brand Awareness*

A major Canadian coffee shop chain is trying to break into the U.S. Market. They assess the following variables twice on a random sample of coffee-drinking adults. The two measurements of each variable are conducted at different times by different interviewers asking somewhat different questions, in such a way that the errors of measurement may be assumed independent. The variables are

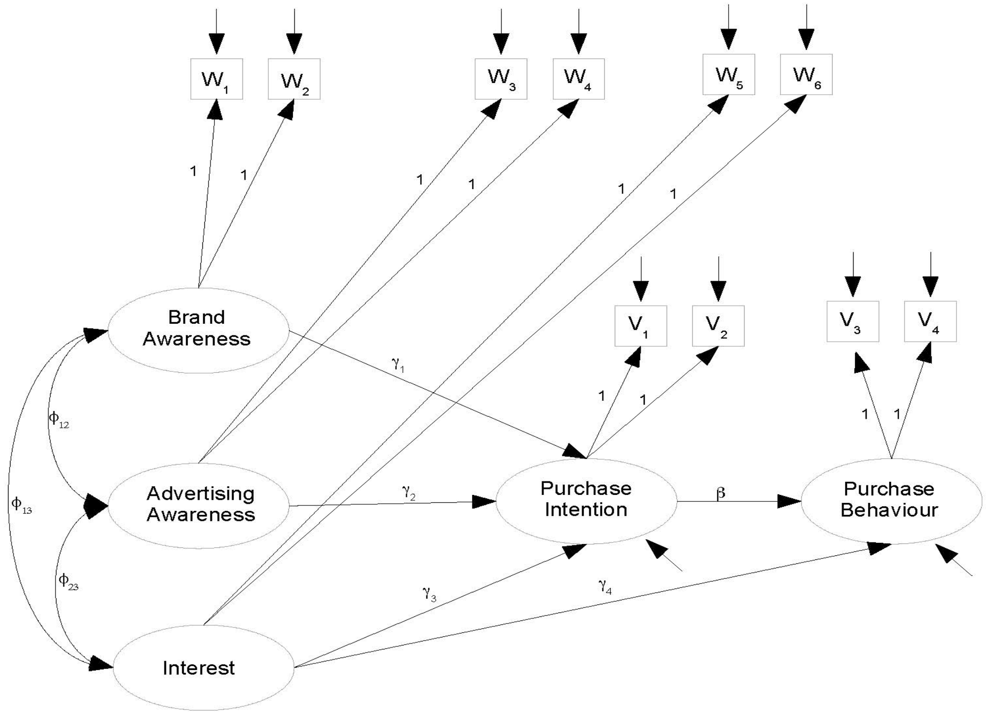
- Brand Awareness: Familiarity with the coffee shop chain
- Advertising Awareness: Recall for advertising of the coffee shop chain
- Interest in the product category: Mostly this was how much they say they like doughnuts.
- Purchase Intention: Expressed willingness to go to an outlet of the coffeeshop chain and make an order.
- Purchase behaviour: Reported dollars spent at the chain during the 2 months following the interview, based on a later telephone interview.

All variables were measured on a scale from 0 to 100 except purchase behaviour, which is in dollars.



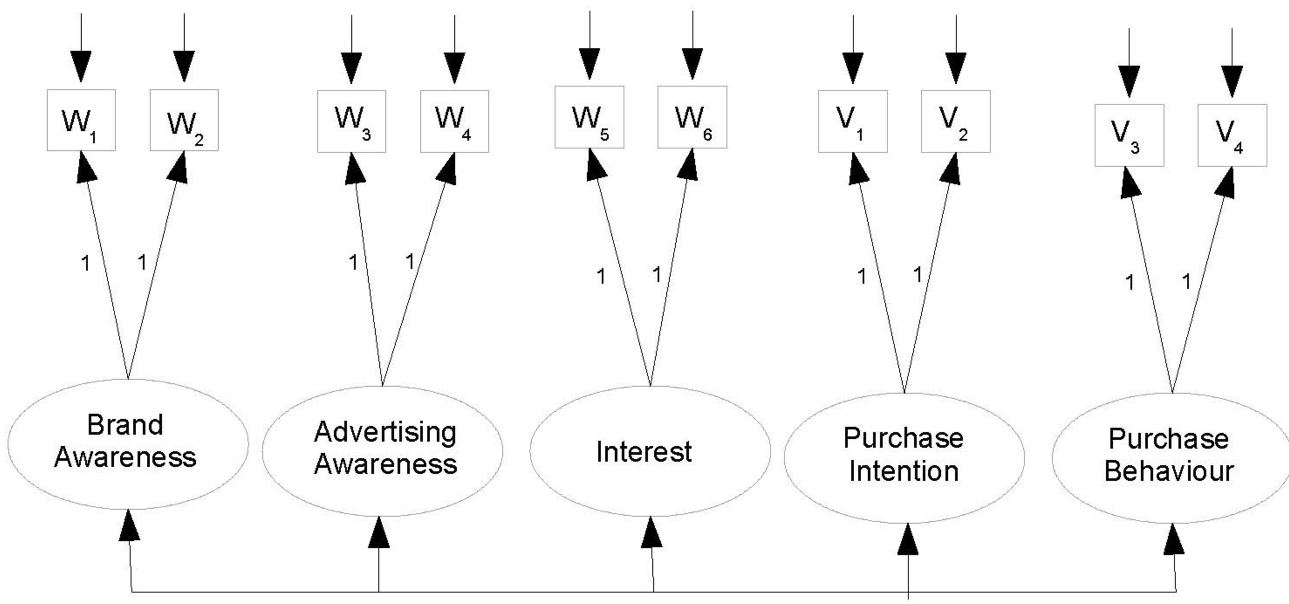
*This handout was prepared by Jerry Brunner, Department of Statistical Sciences, University of Toronto. The OpenOffice.org document is available from the course website:
http://www.utstat.toronto.edu/brunner/oldclass/431s23

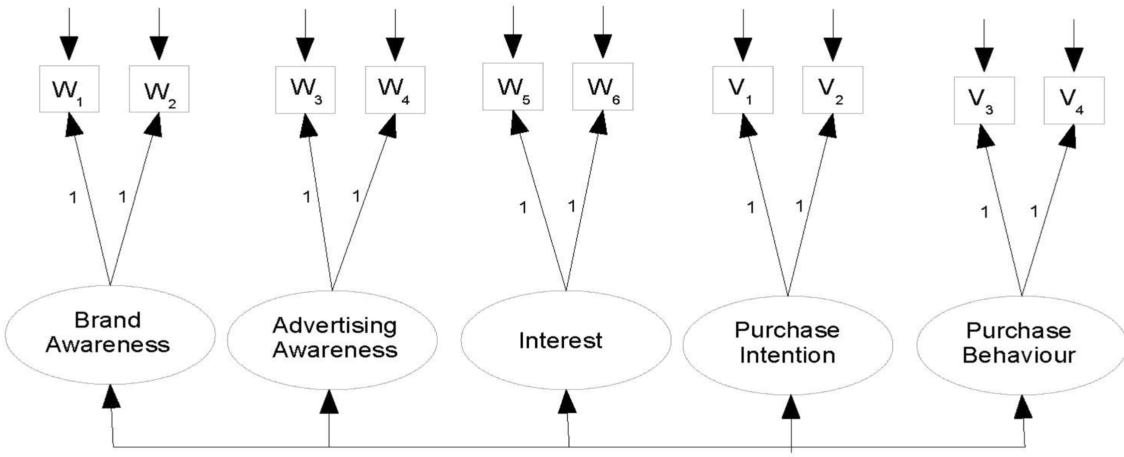
```
> rm(list=ls()); options(scipen=999)
> # install.packages("lavaan", dependencies = TRUE) # Only need to do this once
> library(lavaan)
This is lavaan 0.6-11
lavaan is FREE software! Please report any bugs.
>
> coffee = read.table("http://www.utstat.toronto.edu/brunner/openSEM/data/timmy1.data.txt")
> head(coffee)
  w1 w2 w3 w4 w5 w6 v1 v2 v3 v4
1 40 23 26 21 48 38 22 22 15 15
2 45 24 29 23 49 48 26 13  8 13
3 29 21 21 13 42 37 18 12 13 13
4 38 26 18 19 47 42 20  9 12 10
5 47 31 30 18 48 52 26 16 22 16
6 31 24 18 13 39 40 20 12 16 18
>
> # Observed variables
> #   w1 = Brand Awareness 1
> #   w2 = Brand Awareness 2
> #   w3 = Ad Awareness 1
> #   w4 = Ad Awareness 2
> #   w5 = Interest 1
> #   w6 = Interest 2
> #   v1 = Purchase Intention 1
> #   v2 = Purchase Intention 2
> #   v3 = Purchase Behaviour 1
> #   v4 = Purchase Behaviour 2
> # Latent variables
> #   L_BrAw  = True brand awareness
> #   L_AdAw  = True advertising awareness
> #   L_Inter = True interest in the product category
> #   L_PI    = True purchase intention
> #   L_PBeh  = True purchase behaviour
>
```




```
> torus1 =
+ '
+ # Latent variable model
+     L_PI ~ gamma1*L_BrAw + gamma2*L_AdAw + gamma3*L_Inter
+     L_PBeh    ~ gamma4*L_Inter + beta*L_PI
+ # Measurement model (simple double measurement)
+     L_BrAw  =~ 1*w1 + 1*w2
+     L_AdAw  =~ 1*w3 + 1*w4
+     L_Inter =~ 1*w5 + 1*w6
+     L_PI    =~ 1*v1 + 1*v2
+     L_PBeh  =~ 1*v3 + 1*v4
+ # Variances  and covariances
+     # Exogenous latent variables
+         L_BrAw ~~ phi11*L_BrAw   # Var(L_BrAw)         = phi11
+         L_BrAw ~~ phi12*L_AdAw   # Cov(L_BrAw,L_AdAw)  = phi12
+         L_BrAw ~~ phi13*L_Inter  # Cov(L_BrAw,L_Inter) = phi13
+         L_AdAw ~~ phi22*L_AdAw   # Var(L_AdAw)         = phi22
+         L_AdAw ~~ phi23*L_Inter  # Cov(L_AdAw,L_Inter) = phi23
+         L_Inter ~~ phi33*L_Inter # Var(L_Inter)        = phi33
+     # Errors in the latent model (epsilons)
+         L_PI ~~ psi1*L_PI        # Var(epsilon1) = psi1
+         L_PBeh ~~ psi2*L_PBeh    # Var(epsilon2) = psi2
+     # Measurement errors
+         w1 ~~ omega1*w1          # Var(e1)  = omega1
+         w2 ~~ omega2*w2          # Var(e2)  = omega2
+         w3 ~~ omega3*w3          # Var(e3)  = omega3
+         w4 ~~ omega4*w4          # Var(e4)  = omega4
+         w5 ~~ omega5*w5          # Var(e5)  = omega5
+         w6 ~~ omega6*w6          # Var(e6)  = omega6
+         v1 ~~ omega7*v1          # Var(e7)  = omega7
+         v2 ~~ omega8*v2          # Var(e8)  = omega8
+         v3 ~~ omega9*v3          # Var(e9)  = omega9
+         v4 ~~ omega10*v4         # Var(e10) = omega10
+ # Bounds (Variances are positive)
+     phi11 > 0; phi22 > 0; phi33 > 0
+     psi1 > 0; psi2 > 0
+     omega1 > 0; omega2 > 0; omega3 > 0; omega4 > 0; omega5 > 0
+     omega6 > 0; omega7 > 0; omega8 > 0; omega9 > 0; omega10 > 0
+ ' # End of model torus1
```

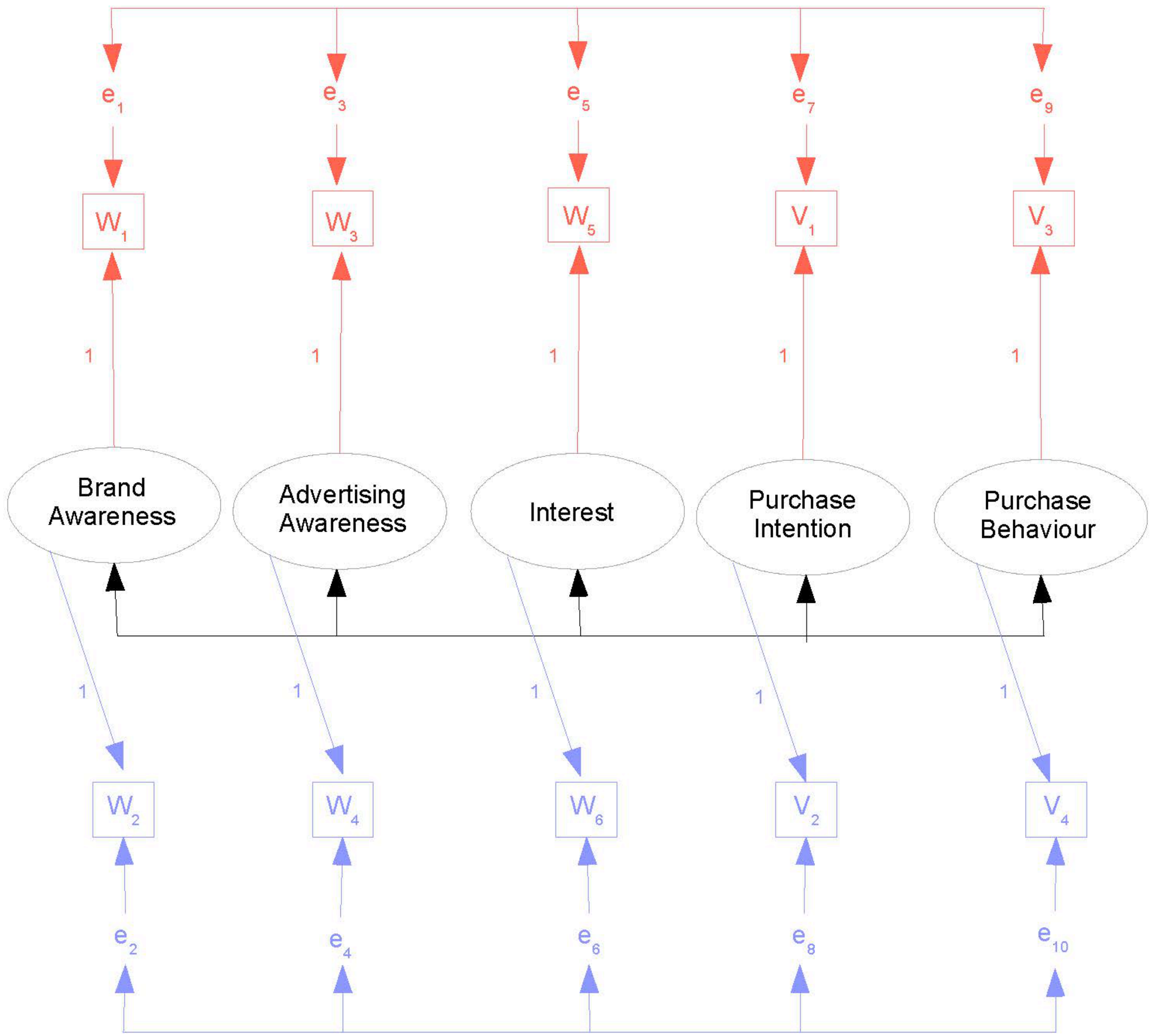
```
> fit1 = lavaan(torus1, data=coffee)
> show(fit1)
lavaan 0.6-11 ended normally after 113 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                        23
  Number of inequality constraints                  15

  Number of observations                           200

Model Test User Model:

  Test statistic                                77.752
  Degrees of freedom                                32
  P-value (Chi-square)                           0.000

>
> # It did not fit, and matched SAS.
> # Split the problem up into parts. Look first at the measurement model.
>
```






```
> torus2 =
+ '
+ # Measurement model (still simple double measurement)
+     L_BrAw  =~ 1*w1 + 1*w2
+     L_AdAw  =~ 1*w3 + 1*w4
+     L_Inter =~ 1*w5 + 1*w6
+     L_PI    =~ 1*v1 + 1*v2
+     L_PBeh  =~ 1*v3 + 1*v4
+ # Variances and covariances
+     # Latent variables
+         L_BrAw ~~ phi11*L_BrAw      # Var(L_BrAw)           = phi11
+         L_BrAw ~~ phi12*L_AdAw      # Cov(L_BrAw, L_AdAw)  = phi12
+         L_BrAw ~~ phi13*L_Inter     # Cov(L_BrAw, L_Inter) = phi13
+         L_BrAw ~~ phi14*L_PI        # Cov(L_BrAw, L_PI)    = phi14
+         L_BrAw ~~ phi15*L_PBeh      # Cov(L_BrAw, L_PBeh)  = phi15
+
+         L_AdAw ~~ phi22*L_AdAw      # Var(L_AdAw)           = phi22
+         L_AdAw ~~ phi23*L_Inter     # Cov(L_AdAw, L_Inter) = phi23
+         L_AdAw ~~ phi24*L_PI        # Cov(L_AdAw, L_PI)    = phi24
+         L_AdAw ~~ phi25*L_PBeh      # Cov(L_AdAw, L_PBeh)  = phi25
+
+         L_Inter ~~ phi33*L_Inter    # Var(L_Inter)          = phi33
+         L_Inter ~~ phi34*L_PI       # Cov(L_Inter, L_PI)   = phi34
+         L_Inter ~~ phi35*L_PBeh     # Cov(L_Inter, L_PBeh) = phi35
+
+         L_PI ~~ phi44*L_PI          # Var(L_PI)             = phi44
+         L_PI ~~ phi45*L_PBeh        # Cov(L_PI, L_PBeh)    = phi45
+
+         L_PBeh ~~ phi55*L_PBeh      # Var(L_PBeh)           = phi55
+     # Measurement errors
+         w1 ~~ omega1*w1          # Var(e1)  = omega1
+         w2 ~~ omega2*w2          # Var(e2)  = omega2
+         w3 ~~ omega3*w3          # Var(e3)  = omega3
+         w4 ~~ omega4*w4          # Var(e4)  = omega4
+         w5 ~~ omega5*w5          # Var(e5)  = omega5
+         w6 ~~ omega6*w6          # Var(e6)  = omega6
+         v1 ~~ omega7*v1          # Var(e7)  = omega7
+         v2 ~~ omega8*v2          # Var(e8)  = omega8
+         v3 ~~ omega9*v3          # Var(e9)  = omega9
+         v4 ~~ omega10*v4         # Var(e10) = omega10
+ # Bounds (Variances are positive)
+     phi11 > 0; phi22 > 0; phi33 > 0; phi44 > 0; phi55 > 0
+     omega1 > 0; omega2 > 0; omega3 > 0; omega4 > 0; omega5 > 0
+     omega6 > 0; omega7 > 0; omega8 > 0; omega9 > 0; omega10 > 0
+ ' # End of model torus2
```

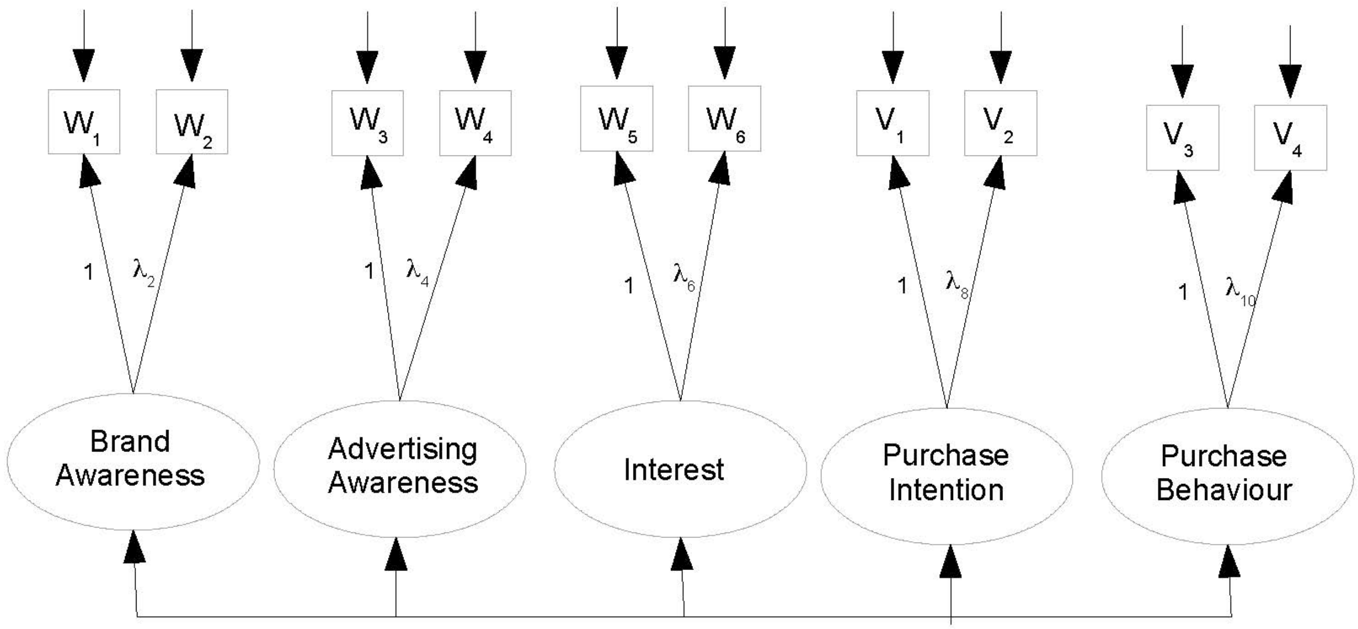
```
> fit2 = lavaan(torus2, data=coffee)
> show(fit2)

lavaan 0.6-11 ended normally after 124 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                        25
  Number of inequality constraints                  15

  Number of observations                           200

Model Test User Model:

  Test statistic                                76.380
  Degrees of freedom                                30
  P-value (Chi-square)                           0.000

>
> # Specify the model more briefly, without all those variances and covariances.
> torus2b =
+ '
+ # Measurement model (still simple double measurement)
+     L_BrAw  =~ 1*w1 + 1*w2
+     L_AdAw  =~ 1*w3 + 1*w4
+     L_Inter =~ 1*w5 + 1*w6
+     L_PI    =~ 1*v1 + 1*v2
+     L_PBeh  =~ 1*v3 + 1*v4
+ ' # End of model torus2b
> fit2b = cfa(torus2b, data=coffee) # Using cfa (confirmatory factor analysis)
>
> show(fit2b)

lavaan 0.6-11 ended normally after 139 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                        25

  Number of observations                           200

Model Test User Model:

  Test statistic                                76.380
  Degrees of freedom                                30
  P-value (Chi-square)                           0.000

> # The measurement model does not fit. Try a true double measurement model, allowing
covariances within sets.
```



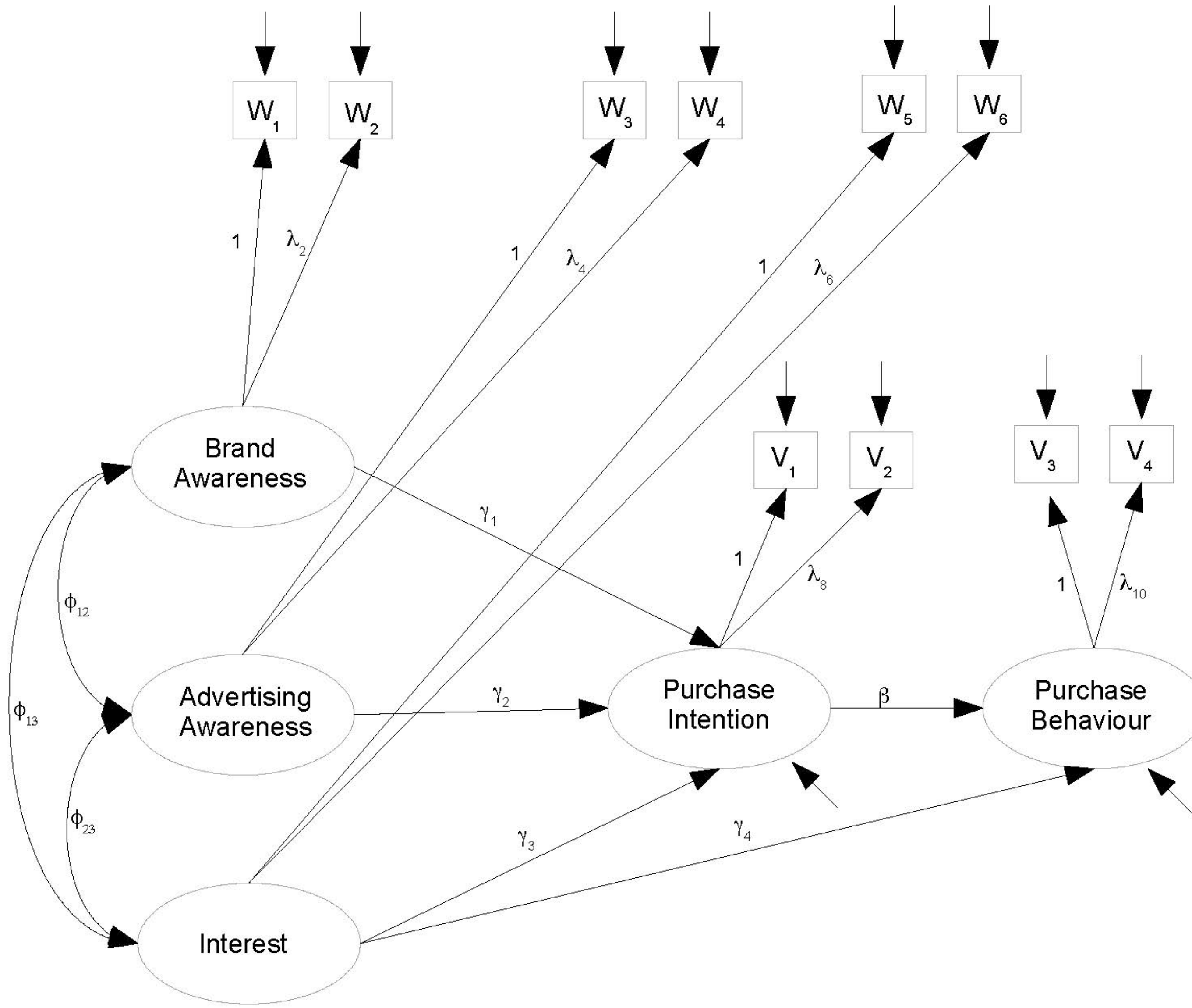

```
> torus3 =
+ '
+ # Measurement model (still simple double measurement)
+     L_BrAw  =~ 1*w1 + 1*w2
+     L_AdAw  =~ 1*w3 + 1*w4
+     L_Inter =~ 1*w5 + 1*w6
+     L_PI    =~ 1*v1 + 1*v2
+     L_PBeh  =~ 1*v3 + 1*v4
+ # Add covariances between measurement error terms, without naming them
+     w1 ~~ w3; w1 ~~ w5; w1 ~~ v1; w1 ~~ v3
+               w3 ~~ w5; w3 ~~ v1; w3 ~~ v3
+                         w5 ~~ v1; w5 ~~ v3
+                                   v1 ~~ v3
+     w2 ~~ w4; w2 ~~ w6; w2 ~~ v2; w2 ~~ v4
+               w4 ~~ w6; w4 ~~ v2; w4 ~~ v4
+                         w6 ~~ v2; w6 ~~ v4
+                                   v2 ~~ v4
+ ' # End of model torus3
```

```
> fit3 = cfa(torus3, data=coffee)

Warning message:
In lav_object_post_check(object) :
  lavaan WARNING: the covariance matrix of the residuals of the observed
                  variables (theta) is not positive definite;
                  use lavInspect(fit, "theta") to investigate.


>
> lavInspect(fit3, "theta")
   w1     w2     w3     w4     w5     w6     v1     v2     v3     v4
w1 10.617
w2  0.000 10.477
w3  2.700  0.000 11.704
w4  0.000 -1.726  0.000 11.263
w5  1.246  0.000  0.475  0.000  8.786
w6  0.000 -3.239  0.000 -1.904  0.000  5.053
v1  3.208  0.000  2.999  0.000  3.933  0.000 13.013
v2  0.000 -2.484  0.000 -1.490  0.000 -3.382  0.000  6.854
v3  0.555  0.000 -0.485  0.000  1.049  0.000  0.875  0.000  4.699
v4  0.000 -1.408  0.000 -1.756  0.000 -0.663  0.000 -1.499  0.000  3.911

> eigen(lavInspect(fit3, "theta"))$values

 [1] 19.183427 12.752307 11.726917 10.823940  9.555380  8.270761  6.184626  4.599928  4.356615
[10] -1.076578

> # Maybe model is wrong, or maybe bad starting values.
> # Obtain MOM estimates for use as starting values.
>
> d1 = as.matrix(coffee[,c(1,3,5,7,9)]) # Measurement set one
> d2 = as.matrix(coffee[,c(2,4,6,8,10)]) # Measurement set two
> Phi_hat = cov(d1,d2); Phi_hat
          w2        w4        w6        v2        v4
w1 10.186131 6.670427 15.123116 11.928618  8.162688
w3  6.655075 8.684598 12.766332 11.339975  6.893844
w5  7.627940 6.536859 16.409548 10.881683  6.290829
v1  8.347940 7.563392 16.891960 15.024598 10.119975
v3  4.674573 3.738015  7.650754  6.998216 17.746859

> # Make it symmetric
> Phi_hat = (Phi_hat + t(Phi_hat) )/2; Phi_hat

          w2        w4        w6        v2        v4
w1 10.186131 6.662751 11.375528 10.138279  6.418631
w3  6.662751 8.684598  9.651595  9.451683  5.315930
w5 11.375528 9.651595 16.409548 13.886822  6.970791
v1 10.138279 9.451683 13.886822 15.024598  8.559095
v3  6.418631 5.315930  6.970791  8.559095 17.746859

> eigen(Phi_hat)$values # Is Phi-hat positive definite?
[1] 50.164191 12.097980  2.925981  1.668071  1.195511

> Omega1_hat = cov(d1) - Phi_hat
> Omega2_hat = cov(d2) - Phi_hat
> eigen(Omega1_hat)$values # Is Omega1_hat positive definite?
[1] 26.402687  9.301147  8.288868  5.106178  2.868356

> eigen(Omega2_hat)$values # Is Omega2_hat positive definite?
[1] 12.867799 11.828405  9.847771  4.712254 -3.393667

> # Method of moments estimate is outside the parameter space
```

Recall that the two measurements of each latent variable are different. One of the interviews is in-person, and the other is by telephone call-back. Maybe they're not really equivalent. Perhaps one in each set (say number two, the call-backs) should have a coefficient not equal to one.




```
> torus4 =
+ '
+ # Measurement model
+     L_BrAw  =~ 1*w1 + lambda2*w2
+     L_AdAw  =~ 1*w3 + lambda4*w4
+     L_Inter =~ 1*w5 + lambda6*w6
+     L_PI    =~ 1*v1 + lambda8*v2
+     L_PBeh  =~ 1*v3 + lambda10*v4
+ ' # End of model torus4
> fit4 = cfa(torus4, data=coffee)
> show(fit4)

lavaan 0.6-11 ended normally after 161 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                        30

  Number of observations                           200

Model Test User Model:

  Test statistic                                17.837
  Degrees of freedom                                25
  P-value (Chi-square)                           0.849
```



W1
W2
W3
W4
W5
W6
1
λ2
1
λ4
1
λ6
V1
V2
V3
V4
Brand Awareness
γ1
1
λ8
1
λ10
φ12
φ13
Advertising Awareness
γ2
Purchase Intention
β
Purchase Behaviour
φ23
γ3
γ4
Interest

```
> # Just edit the measurement model part of model 1
> torus5 =
+ '
+ # Latent variable model
+     L_PI ~ gamma1*L_BrAw + gamma2*L_AdAw + gamma3*L_Inter
+     L_PBeh    ~ gamma4*L_Inter + beta*L_PI
+ # Measurement model
+     L_BrAw  =~ 1*w1 + lambda2*w2
+     L_AdAw  =~ 1*w3 + lambda4*w4
+     L_Inter =~ 1*w5 + lambda6*w6
+     L_PI    =~ 1*v1 + lambda8*v2
+     L_PBeh  =~ 1*v3 + lambda10*v4
+ # Variances  and covariances
+     # Exogenous latent variables
+         L_BrAw ~~ phi11*L_BrAw   # Var(L_BrAw)          = phi11
+         L_BrAw ~~ phi12*L_AdAw   # Cov(L_BrAw,L_AdAw)   = phi12
+         L_BrAw ~~ phi13*L_Inter  # Cov(L_BrAw,L_Inter) = phi13
+         L_AdAw ~~ phi22*L_AdAw   # Var(L_AdAw)          = phi22
+         L_AdAw ~~ phi23*L_Inter  # Cov(L_AdAw,L_Inter) = phi23
+         L_Inter ~~ phi33*L_Inter # Var(L_Inter)         = phi33
+     # Errors in the latent model (epsilons)
+         L_PI ~~ psi1*L_PI        # Var(epsilon1) = psi1
+         L_PBeh ~~ psi2*L_PBeh    # Var(epsilon2) = psi2
+     # Measurement errors
+         w1 ~~ omega1*w1          # Var(e1)  = omega1
+         w2 ~~ omega2*w2          # Var(e2)  = omega2
+         w3 ~~ omega3*w3          # Var(e3)  = omega3
+         w4 ~~ omega4*w4          # Var(e4)  = omega4
+         w5 ~~ omega5*w5          # Var(e5)  = omega5
+         w6 ~~ omega6*w6          # Var(e6)  = omega6
+         v1 ~~ omega7*v1          # Var(e7)  = omega7
+         v2 ~~ omega8*v2          # Var(e8)  = omega8
+         v3 ~~ omega9*v3          # Var(e9)  = omega9
+         v4 ~~ omega10*v4         # Var(e10) = omega10
+ # Bounds (Variances are positive)
+     phi11 > 0; phi22 > 0; phi33 > 0
+     psi1 > 0; psi2 > 0
+     omega1 > 0; omega2 > 0; omega3 > 0; omega4 > 0; omega5 > 0
+     omega6 > 0; omega7 > 0; omega8 > 0; omega9 > 0; omega10 > 0
+ ' # End of model torus5
> fit5 = lavaan(torus5, data=coffee)

Warning messages:

1: In lav_model_vcov(lavmodel = lavmodel, lavsamplestats = lavsamplestats,  :
  lavaan WARNING:
    Could not compute standard errors! The information matrix could
    not be inverted. This may be a symptom that the model is not
    identified.
2: In lav_object_post_check(object) :
  lavaan WARNING: covariance matrix of latent variables
                is not positive definite;
                use lavInspect(fit, "cov.lv") to investigate.
```

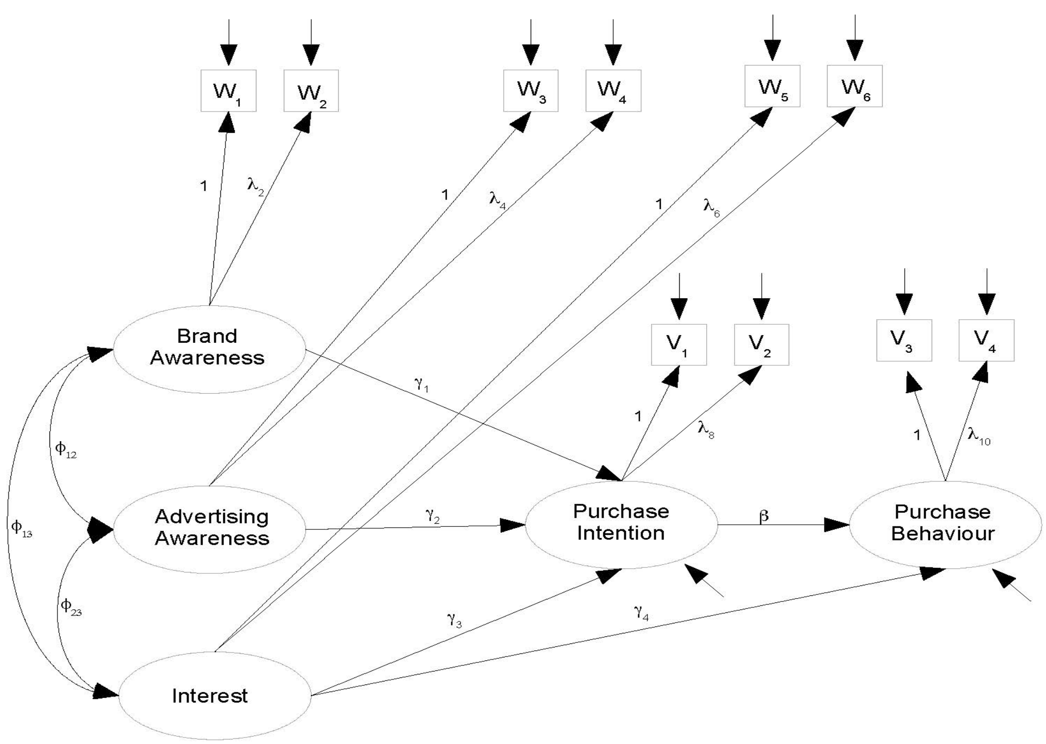
```
> summary(fit5)
lavaan 0.6-11 ended normally after 2096 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                        28
  Number of inequality constraints                  15

  Number of observations                           200

Model Test User Model:

  Test statistic                                31.127
  Degrees of freedom                                27
  P-value (Chi-square)                           0.266

Parameter Estimates:

  Standard errors                             Standard
  Information                                 Expected
  Information saturated (h1) model          Structured

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  L_BrAw =~
    w1                1.000
    w2       (lmb2)   0.535       NA
  L_AdAw =~
    w3                1.000
    w4       (lmb4)   0.552       NA
  L_Inter =~
    w5                1.000
    w6       (lmb6)   1.094       NA
  L_PI =~
    v1                1.000
    v2       (lmb8)   0.708       NA
  L_PBeh =~
    v3                1.000
    v4       (lm10)   1.034       NA

(to be continued)
```

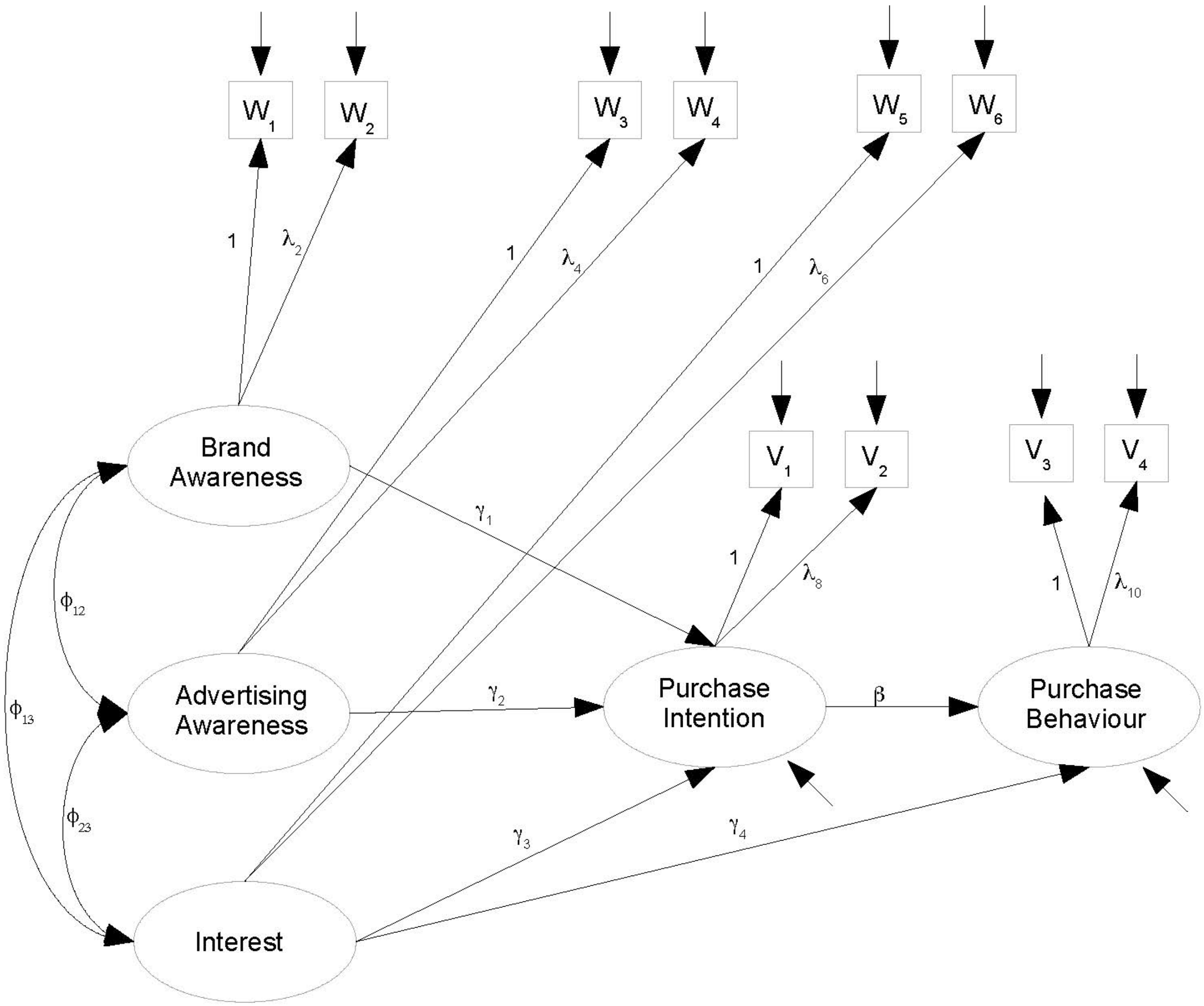
Comparing the estimates from the good measurement model,

```
> coef(fit4)
       lambda2          lambda4         lambda6         lambda8         lambda10
         0.530            0.543           1.090           0.708            1.029
        w1~~w1           w2~~w2          w3~~w3          w4~~w4           w5~~w5
         5.106           12.955           7.034          13.401            6.205
        w6~~w6           v1~~v1          v2~~v2          v3~~v3           v4~~v4
         6.134            8.322          10.301           4.440            3.993
  L_BrAw~~L_BrAw   L_AdAw~~L_AdAw L_Inter~~L_Inter      L_PI~~L_PI   L_PBeh~~L_PBeh
        19.135           15.914          14.980          21.128           17.155
  L_BrAw~~L_AdAw  L_BrAw~~L_Inter     L_BrAw~~L_PI  L_BrAw~~L_PBeh  L_AdAw~~L_Inter
        12.297           13.502          16.248           7.883           11.306
    L_AdAw~~L_PI   L_AdAw~~L_PBeh    L_Inter~~L_PI L_Inter~~L_PBeh      L_PI~~L_PBeh
        15.070            6.144          15.564           6.533            9.619
```

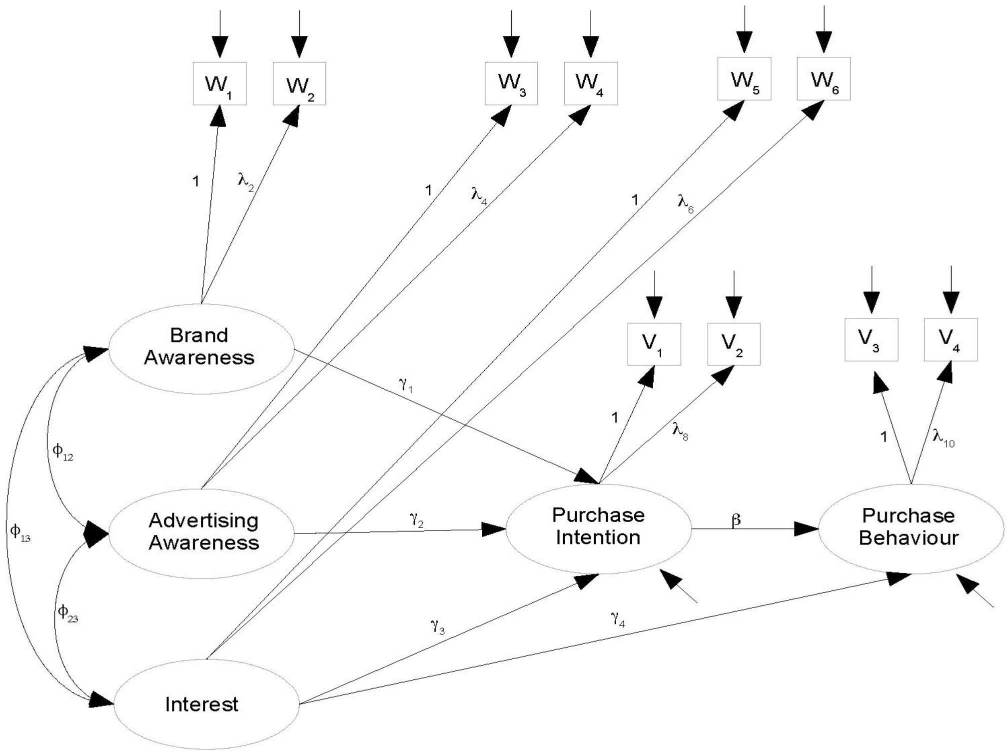
Continuing to look at the output of `summary(fit5)`,

```
Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  L_PI ~
    L_BrAw  (gmm1)   47.719       NA
    L_AdAw  (gmm2) -156.406       NA
    L_Inter (gmm3)   80.361       NA
  L_PBeh ~
    L_Inter (gmm4)   -0.156       NA
    L_PI    (beta)    0.570       NA

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  L_BrAw ~~
    L_AdAw  (ph12)   12.498       NA
    L_Inter (ph13)   13.407       NA
  L_AdAw ~~
    L_Inter (ph23)   11.621       NA

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
    L_BrAw  (ph11)   18.730       NA
    L_AdAw  (ph22)    9.691       NA
    L_Inter (ph33)   14.851       NA
   .L_PI    (psi1)  260.320       NA
   .L_PBeh  (psi2)   12.623       NA
   .w1      (omg1)    5.511       NA
   .w2      (omg2)   12.959       NA
   .w3      (omg3)   13.263       NA
   .w4      (omg4)   15.139       NA
   .w5      (omg5)    6.335       NA
   .w6      (omg6)    6.158       NA
   .v1      (omg7)    8.341       NA
   .v2      (omg8)   10.301       NA
   .v3      (omg9)    4.524       NA
   .v4      (om10)    3.903       NA
```






```
> lavInspect(fit5, "cov.lv")
        L_BrAw L_AdAw L_Intr L_PI   L_PBeh
L_BrAw  18.730
L_AdAw  12.498  9.691
L_Inter 13.407 11.621 14.851
L_PI    16.411 14.534 15.565 21.059
L_PBeh   7.261  6.469  6.554  9.572 17.054
> lavInspect(fit4, "cov.lv") # True MLE
        L_BrAw L_AdAw L_Intr L_PI   L_PBeh
L_BrAw  19.135
L_AdAw  12.297 15.914
L_Inter 13.502 11.306 14.980
L_PI    16.248 15.070 15.564 21.128
L_PBeh   7.883  6.144  6.533  9.619 17.155
>
> # Believe the values from fit4. They are the real MLEs
```




```
# Based on the sub-model y1 = gamma^t x + epsilon1 (details omitted)
# Get estimates of gamma and psi1 = Var(epsilon1)

>
> # The names of all these quantities should include "hat."
> Phi = lavInspect(fit4, "cov.lv")
> Phix = Phi[1:3,1:3]; Phix
           L_BrAw    L_AdAw  L_Inter
L_BrAw  19.13510 12.29660 13.50213
L_AdAw  12.29660 15.91372 11.30579
L_Inter 13.50213 11.30579 14.98033
> Phixy = as.matrix(Phi[1:3,4]); Phixy
            [,1]
L_BrAw  16.24761
L_AdAw  15.07005
L_Inter 15.56443
> gamma = t(Phixy) %*% solve(Phix); gamma
         L_BrAw    L_AdAw   L_Inter
[1,] 0.1996458 0.3932861 0.5622287
> psi1 = Phi[4,4] - as.numeric(gamma %*% Phix %*% t(gamma)); psi1
[1] 3.206661
>
> # These numbers are much more reasonable. See if I can get away with specifying just 10
starting values. Drop the inequality constraints too, since lavaan will issue a warning if
any variance estimate is negative.
```

```
>
> torus6 =
+ '
+ # Latent variable model
+     L_PI ~ gamma1*L_BrAw + start(0.1996458)*L_BrAw +
+     gamma2*L_AdAw        + start(0.3932861)*L_AdAw +
+     gamma3*L_Inter       + start(0.5622287)*L_Inter
+     L_PBeh    ~ gamma4*L_Inter + beta*L_PI
+ # Measurement model
+     L_BrAw  =~ 1*w1 + lambda2*w2
+     L_AdAw  =~ 1*w3 + lambda4*w4
+     L_Inter =~ 1*w5 + lambda6*w6
+     L_PI    =~ 1*v1 + lambda8*v2
+     L_PBeh  =~ 1*v3 + lambda10*v4
+ # Variances  and covariances
+     # Exogenous latent variables
+         L_BrAw ~~ phi11*L_BrAw   + start(19.13510)*L_BrAw  # Var(L_BrAw)         = phi11
+         L_BrAw ~~ phi12*L_AdAw   + start(12.29660)*L_AdAw  # Cov(L_BrAw,L_AdAw)  = phi12
+         L_BrAw ~~ phi13*L_Inter  + start(13.50213)*L_Inter # Cov(L_BrAw,L_Inter) = phi13
+         L_AdAw ~~ phi22*L_AdAw   + start(15.91372)*L_AdAw  # Var(L_AdAw)         = phi22
+         L_AdAw ~~ phi23*L_Inter  + start(11.30579)*L_Inter # Cov(L_AdAw,L_Inter) = phi23
+         L_Inter ~~ phi33*L_Inter + start(14.98033)*L_Inter # Var(L_Inter)        = phi33
+     # Errors in the latent model (epsilons)
+         L_PI ~~ psi1*L_PI + start(3.206661)*L_PI   # Var(epsilon1) = psi1
+         L_PBeh ~~ psi2*L_PBeh                      # Var(epsilon2) = psi2
+     # Measurement errors
+         w1 ~~ omega1*w1          # Var(e1)  = omega1
+         w2 ~~ omega2*w2          # Var(e2)  = omega2
+         w3 ~~ omega3*w3          # Var(e3)  = omega3
+         w4 ~~ omega4*w4          # Var(e4)  = omega4
+         w5 ~~ omega5*w5          # Var(e5)  = omega5
+         w6 ~~ omega6*w6          # Var(e6)  = omega6
+         v1 ~~ omega7*v1          # Var(e7)  = omega7
+         v2 ~~ omega8*v2          # Var(e8)  = omega8
+         v3 ~~ omega9*v3          # Var(e9)  = omega9
+         v4 ~~ omega10*v4         # Var(e10) = omega10
+ ' # End of model torus6
> fit6 = lavaan(torus6, data=coffee)
> fit6

lavaan 0.6-11 ended normally after 108 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                        28

  Number of observations                           200

Model Test User Model:

  Test statistic                                18.962
  Degrees of freedom                                27
  P-value (Chi-square)                           0.871
```

```
> summary(fit6)
lavaan 0.6-11 ended normally after 108 iterations

  Estimator                                         ML
  Optimization method                           NLMINB
  Number of model parameters                        28

  Number of observations                           200

Model Test User Model:

  Test statistic                                18.962
  Degrees of freedom                                27
  P-value (Chi-square)                           0.871

Parameter Estimates:

  Standard errors                             Standard
  Information                                 Expected
  Information saturated (h1) model          Structured

Latent Variables:
                   Estimate  Std.Err  z-value  P(>|z|)
  L_BrAw =~
    w1                1.000
    w2      (lmb2)    0.528    0.077    6.861    0.000
  L_AdAw =~
    w3                1.000
    w4      (lmb4)    0.543    0.090    6.013    0.000
  L_Inter =~
    w5                1.000
    w6      (lmb6)    1.092    0.081   13.528    0.000
  L_PI =~
    v1                1.000
    v2      (lmb8)    0.707    0.066   10.745    0.000
  L_PBeh =~
    v3                1.000
    v4      (lm10)    1.040    0.110    9.457    0.000

Regressions:
                   Estimate  Std.Err  z-value  P(>|z|)
  L_PI ~
    L_BrAw  (gmm1)    0.229    0.145    1.581    0.114
    L_AdAw  (gmm2)    0.369    0.161    2.285    0.022
    L_Inter (gmm3)    0.553    0.170    3.253    0.001
  L_PBeh ~
    L_Inter (gmm4)   -0.129    0.257   -0.502    0.615
    L_PI    (beta)    0.546    0.224    2.438    0.015

Covariances:
                   Estimate  Std.Err  z-value  P(>|z|)
  L_BrAw ~~
    L_AdAw  (ph12)   12.301    1.864    6.598    0.000
    L_Inter (ph13)   13.480    1.831    7.360    0.000
  L_AdAw ~~
    L_Inter (ph23)   11.312    1.694    6.679    0.000

Variances:
                   Estimate  Std.Err  z-value  P(>|z|)
    L_BrAw  (ph11)   19.200    3.110    6.174    0.000
    L_AdAw  (ph22)   15.910    3.033    5.246    0.000
    L_Inter (ph33)   14.961    2.153    6.949    0.000
   .L_PI    (psi1)    3.301    1.340    2.463    0.014
   .L_PBeh  (psi2)   12.620    2.097    6.019    0.000
   .w1      (omg1)    5.041    2.075    2.430    0.015
   .w2      (omg2)   12.974    1.413    9.179    0.000
   .w3      (omg3)    7.038    2.218    3.172    0.002
```

```
    .w4       (omg4)   13.400    1.477    9.074    0.000
    .w5       (omg5)    6.224    0.960    6.484    0.000
    .w6       (omg6)    6.098    1.063    5.735    0.000
    .v1       (omg7)    8.280    1.479    5.598    0.000
    .v2       (omg8)   10.299    1.215    8.477    0.000
    .v3       (omg9)    4.612    1.682    2.742    0.006
    .v4       (om10)    3.809    1.789    2.129    0.033

> parTable(fit6)
   id     lhs op     rhs user block group free ustart exo    label plabel  start    est    se
1   1    L_PI  ~  L_BrAw    1     1     1    1  0.200   0   gamma1   .p1.  0.200  0.229 0.145
2   2    L_PI  ~  L_AdAw    1     1     1    2  0.393   0   gamma2   .p2.  0.393  0.369 0.161
3   3    L_PI  ~ L_Inter    1     1     1    3  0.562   0   gamma3   .p3.  0.562  0.553 0.170
4   4  L_PBeh  ~ L_Inter    1     1     1    4     NA   0   gamma4   .p4.  0.000 -0.129 0.257
5   5  L_PBeh  ~    L_PI    1     1     1    5     NA   0     beta   .p5.  0.000  0.546 0.224
6   6  L_BrAw =~      w1    1     1     1    0  1.000   0            .p6.  1.000  1.000 0.000
7   7  L_BrAw =~      w2    1     1     1    6     NA   0  lambda2   .p7.  0.476  0.528 0.077
8   8  L_AdAw =~      w3    1     1     1    0  1.000   0            .p8.  1.000  1.000 0.000
9   9  L_AdAw =~      w4    1     1     1    7     NA   0  lambda4   .p9.  0.421  0.543 0.090
10 10 L_Inter =~      w5    1     1     1    0  1.000   0           .p10.  1.000  1.000 0.000
11 11 L_Inter =~      w6    1     1     1    8     NA   0  lambda6  .p11.  0.724  1.092 0.081
12 12    L_PI =~      v1    1     1     1    0  1.000   0           .p12.  1.000  1.000 0.000
13 13    L_PI =~      v2    1     1     1    9     NA   0  lambda8  .p13.  0.594  0.707 0.066
14 14  L_PBeh =~      v3    1     1     1    0  1.000   0           .p14.  1.000  1.000 0.000
15 15  L_PBeh =~      v4    1     1     1   10     NA   0 lambda10  .p15.  0.807  1.040 0.110
16 16  L_BrAw ~~  L_BrAw    1     1     1   11 19.135   0    phi11  .p16. 19.135 19.200 3.110
17 17  L_BrAw ~~  L_AdAw    1     1     1   12 12.297   0    phi12  .p17. 12.297 12.301 1.864
18 18  L_BrAw ~~ L_Inter    1     1     1   13 13.502   0    phi13  .p18. 13.502 13.480 1.831
19 19  L_AdAw ~~  L_AdAw    1     1     1   14 15.914   0    phi22  .p19. 15.914 15.910 3.033
20 20  L_AdAw ~~ L_Inter    1     1     1   15 11.306   0    phi23  .p20. 11.306 11.312 1.694
21 21 L_Inter ~~ L_Inter    1     1     1   16 14.980   0    phi33  .p21. 14.980 14.961 2.153
22 22    L_PI ~~    L_PI    1     1     1   17  3.207   0     psi1  .p22.  3.207  3.301 1.340
23 23  L_PBeh ~~  L_PBeh    1     1     1   18     NA   0     psi2  .p23.  0.050 12.620 2.097
24 24      w1 ~~      w1    1     1     1   19     NA   0   omega1  .p24. 12.120  5.041 2.075
25 25      w2 ~~      w2    1     1     1   20     NA   0   omega2  .p25.  9.162 12.974 1.413
26 26      w3 ~~      w3    1     1     1   21     NA   0   omega3  .p26. 11.474  7.038 2.218
27 27      w4 ~~      w4    1     1     1   22     NA   0   omega4  .p27.  9.046 13.400 1.477
28 28      w5 ~~      w5    1     1     1   23     NA   0   omega5  .p28. 10.593  6.224 0.960
29 29      w6 ~~      w6    1     1     1   24     NA   0   omega6  .p29. 11.965  6.098 1.063
30 30      v1 ~~      v1    1     1     1   25     NA   0   omega7  .p30. 14.725  8.280 1.479
31 31      v2 ~~      v2    1     1     1   26     NA   0   omega8  .p31. 10.439 10.299 1.215
32 32      v3 ~~      v3    1     1     1   27     NA   0   omega9  .p32. 10.797  4.612 1.682
33 33      v4 ~~      v4    1     1     1   28     NA   0  omega10  .p33. 11.085  3.809 1.789

  > # Likelihood ratio test of
  > # H0: lambda2 = lambda4 = lambda6 = lambda8 = lambda10 = 1
  > anova(fit1,fit6)
  Chi-Squared Difference Test

       Df   AIC   BIC  Chisq Chisq diff Df diff       Pr(>Chisq)
  fit6 27 10947 11039 18.962
  fit1 32 10996 11071 77.752     58.789       5 0.00000000002162 ***
  ---
  Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

  > # Which ones are different from one?
  > pt6 = parTable(fit6); dim(pt6)
  [1] 33 15
  > z = as.numeric( (pt6[,14]-1)/pt6[,15] )
  > # Extract only meaningful z statistics (for lambda_j)
  > z = z[c(7,9,11,13,15)]
  > names(z) = c('lambda2', 'lambda4', 'lambda6', 'lambda8', 'lambda10')
  > z
     lambda2    lambda4    lambda6    lambda8   lambda10
  -6.1368432 -5.0581710  1.1367154 -4.4540676  0.3614714
  > pt6[c(7,9,11,13,15),14] # Corresponding lambda-hats
  [1] 0.5278696 0.5431214 1.0917385 0.7069418 1.0397428
```