STA431S23 Assignment 6

(1)

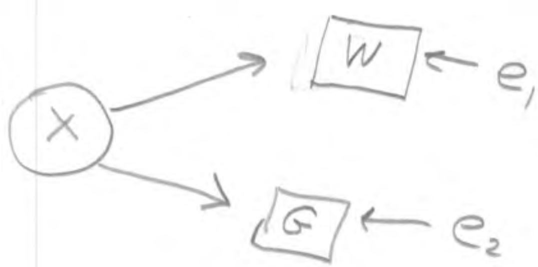
$$\text{Corr}(X, W)^2 = \left(\frac{\text{Cov}(X, \nu + X + e)}{\sqrt{\text{Var}(X)}\,\sqrt{\text{Var}(\nu + X + e)}} \right)^2$$

$$= \left(\frac{\text{Corr}(X, X+e)}{\sqrt{\text{Var}(X)}\,\sqrt{\text{Var}(X+e)}} \right)^2 =$$ Reliability without the intercept

(2) $\text{Corr}(W_1, W_2) = \text{Corr}(\nu_1 + X + e_1, \nu_2 + X + e_2)$

$= \text{Corr}(X + e_1, X + e_2)$ Intercepts don't matter

(3) (a)



(b) $$\text{Corr}(W, G)^2 = \left(\frac{\text{Cov}(W, G)}{\sqrt{\text{Var}(W)\,\text{Var}(G)}} \right)^2$$

$$= \frac{(\text{Cov}(X + e_1, X + e_2))^2}{(\sigma_x^2 + \sigma_1^2)(\sigma_x^2 + \sigma_2^2)} = \frac{(\sigma_x^2)^2}{(\sigma_x^2 + \sigma_1^2)(\sigma_x^2 + \sigma_2^2)}$$

Reliability of W

$$< \frac{\sigma_x^2}{\sigma_x^2 + \sigma_1^2} \iff \frac{\sigma_x^2}{\sigma_x^2 + \sigma_2^2} < 1$$

$$\iff \sigma_x^2 < \sigma_x^2 + \sigma_2^2 \implies \sigma_2^2 > 0 \quad \blacksquare$$

4

(a) $Var(S) = Var(W_1 + W_2) = Var(X + e_1 + X + e_2)$

$= Var(2X + e_1 + e_2) = 4\sigma_x^2 + \sigma_e^2 + \sigma_e^2$

$= 4\sigma_x^2 + 2\sigma_e^2$, so Reliability of $S = W_1 + W_2$

$$= \left(\frac{Cov(X, S)}{\sqrt{Var(X)\, Var(S)}}\right)^2 = \left(\frac{Cov(X, 2X + e_1\, e_2)}{\sqrt{\sigma_x^2 (4\sigma_x^2 + 2\sigma_e^2)}}\right)^2$$

$$= \frac{(2\sigma_x^2 + 0)^2}{\sigma_x^2 (4\sigma_x^2 + 2\sigma_e^2)} = \frac{4\sigma_x^2 \sigma_x^2 \cdot \frac{1}{4\sigma_x^2}}{(4\sigma_x^2\sigma_x^2 + 2\sigma_x^2\sigma_e^2)\frac{1}{4\sigma_x^2}}$$

$$= \frac{\sigma_x^2}{\sigma_x^2 + \frac{1}{2}\sigma_e^2} > \frac{\sigma_x^2}{\sigma_x^2 + \sigma_e^2}$$

(b) $S_n = \sum_{i=1}^{n} W_i$. $Var(S_n) = Var\left(\sum_{i=1}^{n}(X + e_i)\right)$

$$= Var\left(nX + \sum_{i=1}^{n} e_i\right) \overset{ind}{=} n^2\sigma_x^2 + \sum_{i=1}^{n}\sigma_e^2$$

$$= n^2\sigma_x^2 + n\,\sigma_e^2$$

(4b. continued) Reliability of S

$$= \mathrm{Corr}(X,S)^2 = \frac{\left(\mathrm{Cov}(X, \sum_{i=1}^{n}(x+e_i)\right)^2}{\sigma_x^2(n^2\sigma_x^2 + n\sigma_e^2)}$$

$$= \frac{\left(\mathrm{Cov}(X, nX + \sum_{i=1}^{n} e_i)\right)^2}{n^2\sigma_x^2(\sigma_x^2 + \frac{1}{n}\sigma_e^2)}$$

$$= \frac{(n\sigma_x^2 + 0)^2}{n^2\sigma_x^2(\sigma_x^2 + \frac{1}{n}\sigma_e^2)} = \frac{\cancel{n^2\sigma_x^2}\,\sigma_x^2}{\cancel{n^2\sigma_x^2}\,(\sigma_x^2 + \frac{1}{n}\sigma_e^2)}$$

(c) Reliability of $\overline{W}_n = \frac{1}{n}\sum_{i=1}^{n} W_i = \frac{1}{n} S_n$

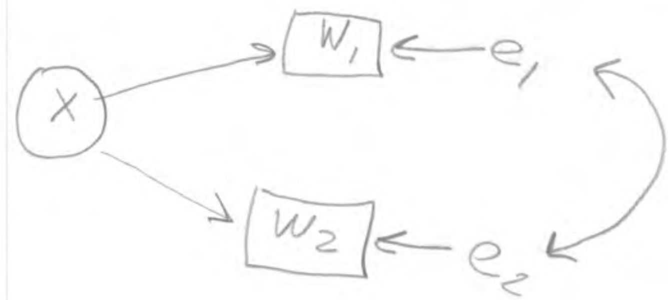
$$= \frac{\left(\mathrm{Cov}(X, \frac{1}{n}S_n)\right)^2}{\mathrm{Var}(X)\,\mathrm{Var}(\frac{1}{n}S_n)} = \frac{\left(\frac{1}{n}\mathrm{Cov}(X, S_n)\right)^2}{\frac{1}{n^2}\mathrm{Var}(X)\,\mathrm{Var}(S_n)}$$

$$= \frac{\cancel{\frac{1}{n^2}}\left(\mathrm{Cov}(X,S_n)\right)^2}{\cancel{\frac{1}{n^2}}\left(\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(S_n)}\right)^2} = \text{Reliability of } S_n$$

$$= \frac{\sigma_x^2}{\sigma_x^2 + \frac{1}{n}\sigma_e^2}$$

d) Reliability $\to 1$ as $n \to \infty$ Longer tests are more reliable

5 (a)



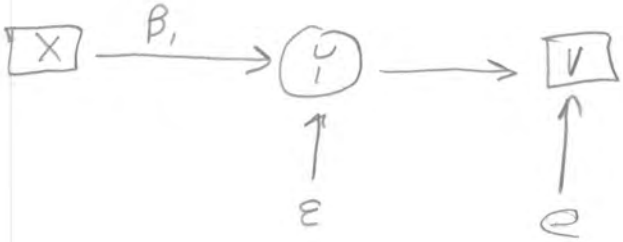
(b) Reliability is $\dfrac{\sigma_x^2}{\sigma_x^2+\sigma_e^2}$

$$Corr(W_1, W_2) = \frac{cov(X+e_1, X+e_2)}{\sqrt{\sigma_x^2+\sigma_e^2}\sqrt{\sigma_x^2+\sigma_e^2}}$$

$$= \frac{cov(X,X)+Cov(e_1,e_2)}{\sigma_x^2+\sigma_e^2} = \frac{\sigma_x^2 + c}{\sigma_x^2+\sigma_e^2} > \frac{\sigma_x^2}{\sigma_x^2+\sigma_e^2}$$

because $c > 0$.

6 (a)



(b) $\theta = (\beta_0, \beta_1, \mu_x, \phi, \omega, \psi)$ 6 parameters

There are 3 σ_{ij} and 2 μ_j for 5 moments

The true model fails the parameter count rule.

(6c)

	x	v
x	σ_x^2	$\beta_1 \sigma_x^2$
v		$\beta_1^2 \sigma_x^2 + \psi + \omega$

$$y = \beta_0 + \beta_1 x + \varepsilon$$
$$v = y + e$$
$$= \beta_0 + \beta_1 x + \varepsilon + e$$
$$Cov(x, v) = \beta_1 \sigma_x^2$$

(d)

$$\hat{\beta}_1 = \frac{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(v_i - \bar{v})}{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{\hat{\sigma}_{xv}}{\hat{\sigma}_x^2} \xrightarrow{p} \frac{\sigma_{xv}}{\sigma_x^2}$$

By LLN and continuous mapping,

$$\frac{\sigma_{xv}}{\sigma_x^2} = \frac{\beta_1 \sigma_x^2}{\sigma_x^2} = \beta_1$$ consistent, Yes

(e) β_1 is identifiable, because otherwise consistent estimation would be impossible

7 (a) $\theta = (\beta, \phi, \psi, \omega)$

(b)

$$\Sigma = \begin{array}{c c} & \begin{array}{cc} w & y \end{array} \\ \begin{array}{c} w \\ y \end{array} & \left(\begin{array}{c|c} \phi + \omega & \beta\phi \\ \hline & \beta^2\phi + \psi \end{array} \right) \end{array}$$

(c) No. There are 4 parameters and three unique moments.

(d) β is identifiable at all (infinitely many) points where $\beta = 0$: $\{(\beta, \phi, \psi, \omega) : \beta = 0, \phi > 0$
$\psi > 0, \omega > 0\}$

Because $\beta = 0$ if and only if $\sigma_{12} = 0$

(e) $\hat{\beta}$ cannot be consistent because β is not identifiable in the whole parameter space.

This is correct, but requires proof. Sorry about that!

(7f)
$$\hat{\beta}_n = \frac{\frac{1}{n}\sum_{i=1}^{n} W_i Y_i}{\frac{1}{n}\sum_{i=1}^{n} W_i^2} = \frac{\hat{\sigma}_{wy}}{\hat{\sigma}_w^2} \xrightarrow{p} \frac{\sigma_{wy}}{\sigma_w^2}$$

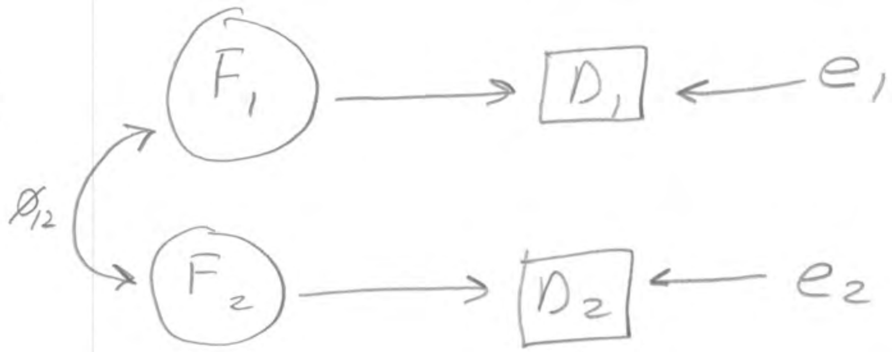
$$= \frac{\beta\phi}{\phi + \omega} = \left(\frac{\phi}{\phi+\omega}\right)\beta$$

(g) $\hat{\beta}_n \xrightarrow{p} \beta$ where $\beta = 0$. This is exactly where β is identifiable.

(h) Try $\hat{\beta}_n / r^2_{wx}$. If r_{wx} is consistent, this estimator will be consistent too, by continuous mapping (and the stack theorem).

8

(a)



(b) $$Corr(D_1, D_2) = \frac{Cov(D_1, D_2)}{SD(D_1)\ SD(D_2)} = \frac{Cov(F_1+e_1, F_2+e_2)}{\sqrt{(\phi_{11}+\omega_1)(\phi_{22}+\omega_2)}}$$

$$= \frac{Cov(F_1, F_2)+0}{\sqrt{(\phi_{11}+\omega_1)(\phi_{22}+\omega_2)}} = \frac{\phi_{12}}{\sqrt{(\phi_{11}+\omega_1)(\phi_{22}+\omega_2)}}$$

And

$$|Corr(D_1, D_2)| = \frac{|\phi_{12}|}{\sqrt{(\phi_{11}+\omega_1)(\phi_{22}+\omega_2)}}$$

$$< \frac{|\phi_{12}|}{\sqrt{\phi_{11}\ \phi_{22}}} = |Corr(F_1, F_2)|$$

Since $\omega_1 > 0$ and $\omega_2 > 0$.

(8c) $$\frac{\text{Corr}(D_1, D_2)}{P_1 \quad P_2} = \frac{\dfrac{\phi_{12}}{\sqrt{(\phi_{11}+w_1)(\phi_{22}+w_2)}}}{\sqrt{\dfrac{\phi_{11}}{\phi_{11}+w_1}}\sqrt{\dfrac{\phi_{22}}{\phi_{22}+w_2}}}$$

$$= \frac{\phi_{12}}{\cancel{\sqrt{\phi_{11}+w_1}}\cancel{\sqrt{\phi_{22}+w_2}}} \cdot \frac{\cancel{\sqrt{\phi_{11}+w_1}}\cancel{\sqrt{\phi_{22}+w_2}}}{\sqrt{\phi_{11}}\sqrt{\phi_{22}}}$$

$$= \frac{\phi_{12}}{\sqrt{\phi_{11}\phi_{22}}} = \text{Corr}(F_1, F_2)$$

(d) $$\frac{0.25}{\sqrt{0.90}\sqrt{0.75}} = 0.304$$

(9) (a) Dividing numerator and denominator by n^2,

$$\hat{\beta}_2 = \frac{\frac{1}{n}\sum_{i=1}^{n} W_i^2 \frac{1}{n}\sum_{i=1}^{n} X_{i2} Y_i - \frac{1}{n}\sum_{i=1}^{n} W_i X_{i2} \frac{1}{n}\sum_{i=1}^{n} W_i Y_i}{\frac{1}{n}\sum_{i=1}^{n} W_i^2 \frac{1}{n}\sum_{i=1}^{n} X_{i2}^2 - \left(\frac{1}{n}\sum_{i=1}^{n} W_i X_{i2}\right)^2}$$

$$\xrightarrow{p} \frac{E(W^2)E(X_2Y) - E(WX_2)E(WY)}{E(W^2)E(X_2^2) - (E(WX_2))^2}$$

Need to calculate two expected values.

$$E(W^2) = Var(W) = Var(X_1) + Var(e) = \phi_{11} + \omega$$

$$E(X_2Y) = E\{X_2(\beta_1 X_1 + \beta_2 X_2 + \varepsilon)\} = \beta_1 E(X_1X_2) + \beta_2 E(X_2^2)$$
$$= \beta_1\phi_{12} + \beta_2\phi_{22}$$

$$E(WX_2) = E\{(X_1+e)X_2\} = E(X_1X_2) = \phi_{12}$$

$$E(WY) = E\{(X_1+e)(\beta_1X_1 + \beta_2X_2 + \varepsilon)\} = \beta_1 E(X_1^2) + \beta_2 E(X_1X_2)$$
$$= \beta_1\phi_{11} + \beta_2\phi_{12}$$

$E(X_2^2) = \phi_{22}$, so

$$\hat{\beta}_2 \xrightarrow{p} \frac{(\phi_{11}+\omega)(\beta_1\phi_{12} + \beta_2\phi_{22}) - \phi_{12}(\beta_1\phi_{11} + \beta_2\phi_{12})}{(\phi_{11}+\omega)\phi_{22} - \phi_{12}^2}$$

$$= \frac{\cancel{\phi_{11}\beta_1\phi_{12}} + \phi_{11}\beta_2\phi_{22} + \omega(\beta_1\phi_{12} + \beta_2\phi_{22}) - \cancel{\phi_{12}\beta_1\phi_{11}} - \phi_{12}^2\beta_2}{(\phi_{11}+\omega)\phi_{22} - \phi_{12}^2}$$

$$= \frac{\beta_2(\phi_{11}\phi_{22} - \phi_{12}^2) + \omega(\beta_1\phi_{12} + \beta_2\phi_{22})}{(\phi_{11}+\omega)\phi_{22} - \phi_{12}^2}$$

(9b) With $w = 0$,

$$\frac{\beta_2(\phi_{11}\phi_{22}-\phi_{12}^2) + w(\beta_1\phi_{12}+\beta_2\phi_{22})}{(\phi_{11}+w)\phi_{22} - \phi_{12}^2}$$

$$= \frac{\beta_2(\phi_{11}\phi_{22}-\phi_{12}^2)}{\phi_{11}\phi_{22}-\phi_{12}^2} = \beta_2$$

(c) with $w > 0$ and $\beta_2 = 0$, target is

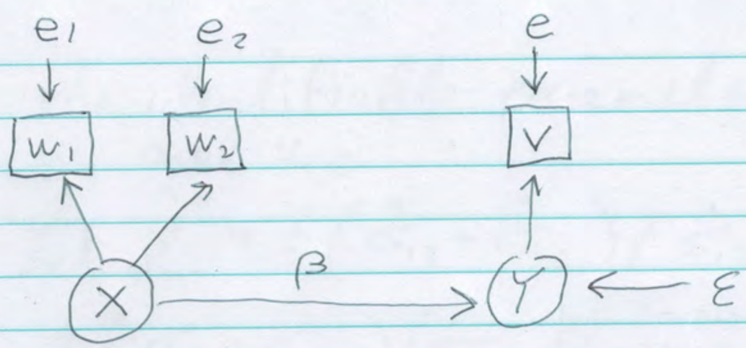
$$\frac{w\beta_1\phi_{12}}{(\phi_{11}+w)\phi_{22} - \phi_{12}^2}$$

No. It's only zero when $\beta_1 = 0$, $\phi_{12} = 0$, or both

(d)

10

(a)



(b) $\theta = (\beta, \phi, \psi, \omega, \omega_1, \omega_2)$

(c) There are 6 parameters and $3(3+1)/2 = 6$ covariance structure equation, so yes.

(d)

	W_1	W_2	V
W_1	$\phi + \omega_1$	ϕ	$\beta\phi$
W_2		$\phi + \omega_2$	$\beta\phi$
V			$\beta^2\phi + \psi + \omega$

$$Var(V) = Var(Y + e) = Var(\beta X + \varepsilon + e) = \beta^2\phi + \psi + \omega$$

$$Cov(W_1, W_2) = Cov(X + e_1, X + e_2) = Cov(X, X) + 0 = \phi$$

$$Cov(W_1, V) = Cov(X + e_1, \beta X + \varepsilon + e) = \beta\phi = Cov(W_2, V)$$

(e) No. The parameters ψ and ω only appear in Σ as $\psi + \omega$. This means that any two parameter vectors $\theta = (\beta, \phi\, \psi, \omega, \omega_1, \omega_2)$ and
$\theta' = (\beta', \phi', \psi', \omega', \omega_1', \omega_2')$ with
$\beta = \beta'$, $\phi = \phi'$, $\omega_1 = \omega_1'$ and $\omega_2 = \omega_2'$ but $\psi \neq \psi'$ & $\omega \neq \omega'$ with $\psi + \omega = \psi' + \omega'$ will yield the same Σ and the same distribution of the observable data.

(Or you could give a numerical example)

(10f) The identifiable parameters are ϕ, β, ω_1 and ω_2

(g) Set $\hat{\beta}_n = \frac{1}{2}(\hat{\sigma}_{13} + \hat{\sigma}_{23}) / \hat{\sigma}_{12}$

$\xrightarrow{p} \frac{1}{2}(\sigma_{13} + \sigma_{23}) / \sigma_{12}$ by consistency of the sample covariance, Slutsky's theorem and continuous mapping

$$\frac{1}{2}(\beta\phi + \beta\phi) / \phi = \frac{1}{2} 2\beta\cancel{\phi}/\cancel{\phi} = \beta$$

(h) Using answer to (g)

$\hat{\beta}_n = \frac{1}{2}(19.85 + 19.00) / 21.39 = 0.908$

(i) Set $c = \psi + w$. Then the parameter vector $(\beta, \phi, c, \omega_1, \omega_2)$ is identifiable.