

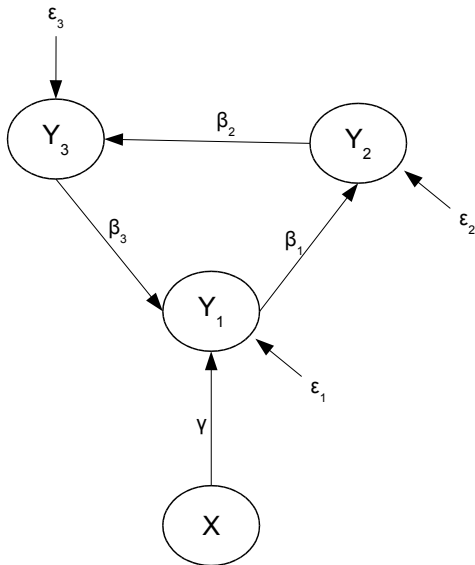
The Pinwheel Model¹

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A Cyclic Model

Not an *Acyclic* model



Model equations

Assume all variances positive etc.

$$Y_1 = \beta_3 Y_3 + \gamma X + \epsilon_1$$

$$Y_2 = \beta_1 Y_1 + \epsilon_2$$

$$Y_3 = \beta_2 Y_2 + \epsilon_3$$

In matrix terms:

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \beta_3 \\ \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} + \begin{pmatrix} \gamma \\ 0 \\ 0 \end{pmatrix} X + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}$$

To get $V(\mathbf{Y})$

$$\begin{aligned}\mathbf{Y} &= \beta\mathbf{Y} + \Gamma\mathbf{X} + \epsilon \\ \Rightarrow \mathbf{Y} - \beta\mathbf{Y} &= \Gamma\mathbf{X} + \epsilon \\ \Rightarrow \mathbf{I}\mathbf{Y} - \beta\mathbf{Y} &= \Gamma\mathbf{X} + \epsilon \\ \Rightarrow (\mathbf{I} - \beta)\mathbf{Y} &= \Gamma\mathbf{X} + \epsilon\end{aligned}$$

$(\mathbf{I} - \beta)^{-1}$ exists when $|\mathbf{I} - \beta| \neq 0$

$$\mathbf{I} - \beta = \begin{pmatrix} 1 & 0 & -\beta_3 \\ -\beta_1 & 1 & 0 \\ 0 & -\beta_2 & 1 \end{pmatrix}$$

Calculate the determinant using Sage

```
sem = 'http://www.utstat.toronto.edu/~brunner/openSEM/sage/sem.sage'  
load(sem)  
B = ZeroMatrix(3,3)  
B[0,2] = var('beta3'); B[1,0] = var('beta1'); B[2,1] = var('beta2')  
ImB = IdentityMatrix(3)-B  
show( ImB.determinant() )
```

$$-\beta_1\beta_2\beta_3 + 1$$

So the inverse will exist unless $\beta_1\beta_2\beta_3 = 1$.

Solve for Y_3

Starting with the model equations

$$Y_1 = \beta_3 Y_3 + \gamma X + \epsilon_1$$

$$Y_2 = \beta_1 Y_1 + \epsilon_2$$

$$Y_3 = \beta_2 Y_2 + \epsilon_3$$

$$\begin{aligned} Y_3 &= \beta_1 \beta_2 \beta_3 Y_3 + \beta_1 \beta_2 \gamma X + \beta_1 \beta_2 \epsilon_1 + \beta_2 \epsilon_2 + \epsilon_3 \\ \Rightarrow Y_3(1 - \beta_1 \beta_2 \beta_3) &= \beta_1 \beta_2 \gamma X + \beta_1 \beta_2 \epsilon_1 + \beta_2 \epsilon_2 + \epsilon_3 \end{aligned}$$

What happens if $(\mathbf{I} - \boldsymbol{\beta})^{-1}$ does not exist (and $\gamma \neq 0$)?

If $\beta_1\beta_2\beta_3 = 1$

Meaning that $(\mathbf{I} - \boldsymbol{\beta})^{-1}$ does not exist

$$Y_3(1 - \beta_1\beta_2\beta_3) = \beta_1\beta_2\gamma X + \beta_1\beta_2\epsilon_1 + \beta_2\epsilon_2 + \epsilon_3$$

$$\Rightarrow 0 = \beta_1\beta_2\gamma X + \beta_1\beta_2\epsilon_1 + \beta_2\epsilon_2 + \epsilon_3$$

$$\Rightarrow E(X \cdot 0) = E(X(\beta_1\beta_2\gamma X + \beta_1\beta_2\epsilon_1 + \beta_2\epsilon_2 + \epsilon_3))$$

$$\Rightarrow 0 = \beta_1\beta_2\gamma E(X^2) + 0$$

$$\Rightarrow \beta_1\beta_2\gamma\phi = 0$$

with β_1, β_2, γ and ϕ all non-zero.

So $\beta_1\beta_2\beta_3 = 1$ contradicts the model.

Under the assumptions of the pinwheel model

- $(\mathbf{I} - \boldsymbol{\beta})^{-1}$ exists.
- $\beta_1\beta_2\beta_3 \neq 1$.
- The surface $\beta_1\beta_2\beta_3 = 1$ forms a *hole* in the parameter space.

Covariance matrix of the factors: Φ

Factors are X, Y_1, Y_2, Y_3

$$\left(\begin{array}{ccc} \phi & & \\ -\frac{\gamma\phi}{\beta_1\beta_2\beta_3-1} & -\frac{\beta_1\gamma\phi}{\beta_1\beta_2\beta_3-1} & -\frac{\beta_1\beta_2\gamma\phi}{\beta_1\beta_2\beta_3-1} \\ \frac{\beta_2^2\beta_3^2\psi_2+\beta_3^2\psi_3+\gamma^2\phi+\psi_1}{(\beta_1\beta_2\beta_3-1)^2} & \frac{\beta_1\beta_3^2\psi_3+\beta_1\gamma^2\phi+\beta_2\beta_3\psi_2+\beta_1\psi_1}{(\beta_1\beta_2\beta_3-1)^2} & \frac{\beta_1\beta_2\gamma^2\phi+\beta_2^2\beta_3\psi_2+\beta_1\beta_2\psi_1+\beta_3\psi_3}{(\beta_1\beta_2\beta_3-1)^2} \\ & \frac{\beta_1^2\beta_3^2\psi_3+\beta_1^2\gamma^2\phi+\beta_1^2\psi_1+\psi_2}{(\beta_1\beta_2\beta_3-1)^2} & \frac{\beta_1^2\beta_2\gamma^2\phi+\beta_1^2\beta_2\psi_1+\beta_1\beta_3\psi_3+\beta_2\psi_2}{(\beta_1\beta_2\beta_3-1)^2} \\ & & \frac{\beta_1^2\beta_2^2\gamma^2\phi+\beta_1^2\beta_2^2\psi_1+\beta_2^2\psi_2+\psi_3}{(\beta_1\beta_2\beta_3-1)^2} \end{array} \right)$$

- $\phi = \phi_{11}$, $\beta_1 = \frac{\phi_{13}}{\phi_{12}}$ and $\beta_2 = \frac{\phi_{14}}{\phi_{13}}$ are easy.
- But then?

Solutions exist provided $\beta_1, \beta_2, \beta_3$ are all non-zero.

Using Sage ...

$$\beta_3 = \frac{\phi_{12}\phi_{13}\phi_{23} - \phi_{13}^2\phi_{22}}{\phi_{12}\phi_{14}\phi_{33} - \phi_{13}\phi_{14}\phi_{23}}$$

$$\gamma = \frac{\phi_{12}^2\phi_{33} - 2\phi_{12}\phi_{13}\phi_{23} + \phi_{13}^2\phi_{22}}{\phi_{11}\phi_{12}\phi_{33} - \phi_{11}\phi_{13}\phi_{23}}$$

$$\psi_3 = \frac{(\phi_{13}\phi_{44} - \phi_{14}\phi_{34})(\phi_{12}^2\phi_{33} - 2\phi_{12}\phi_{13}\phi_{23} + \phi_{13}^2\phi_{22})}{(\phi_{12}\phi_{33} - \phi_{13}\phi_{23})\phi_{12}\phi_{13}}$$

$$\psi_2 = \frac{\phi_{12}^2\phi_{33} - 2\phi_{12}\phi_{13}\phi_{23} + \phi_{13}^2\phi_{22}}{\phi_{12}^2}$$

$$\psi_1 = \beta_1^2\beta_2^2\beta_3^2\phi_{22} - \beta_2^2\beta_3^2\psi_2 - 2\beta_1\beta_2\beta_3\phi_{22} - \beta_3^2\psi_3 - \gamma^2\phi + \phi_{22}$$

Parameters of the pinwheel model are identifiable

- Even though it does not fit any known rules
- And the proof is very difficult.

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<http://www.utstat.toronto.edu/~brunner/oldclass/431s15>