STA 431s09 Assignment 2

Do this assignment in preparation for the quiz on Friday, Jan. 23d. For the first question, you will hand in a one-page printout, which will be worth two marks out of ten. The other questions are practice for the rest of the quiz, and are not to be handed in.

1. Let $X_1, \ldots, X_5$ be a random sample from a Gamma distribution with parameters α and $\beta = 1$. That is, the density is

   $$f(x;\alpha) = \frac{1}{\Gamma(\alpha)} e^{-x} x^{\alpha-1}$$

   for $x > 0$, and zero otherwise. The parameter α is greater than zero.

   The five data values are 2.06, 1.08, 0.96, 1.32, 1.53. Find an approximate numerical value of the maximum likelihood estimate of α. Your final answer is one number.

   You will need to use a computer for this question, which will be worth two marks out of ten on the quiz. You will hand in a one-page printout. On the back, you will write a brief explanation of what you did. Use any software you like. I got the answer to one decimal place of accuracy (good enough for full marks) using Excel in 25 minutes, and to 6 decimal places of accuracy in 10 minutes with R.

2. Let $\mathbf{X}$ and $\mathbf{Y}$ be random matrices of the same dimensions. Show $E(\mathbf{X}+\mathbf{Y}) = E(\mathbf{X}) + E(\mathbf{Y})$. Recall the definition $E(\mathbf{Z}) = [E(Z_{i,j})]$.

3. Let $\mathbf{X}$ be a random matrix, and $\mathbf{B}$ be a matrix of constants. Show $E(\mathbf{XB}) = E(\mathbf{X})\mathbf{B}$. Recall the definition $\mathbf{AB} = [\sum_k a_{i,k} b_{k,j}]$.

4. If the $p \times 1$ random vector $\mathbf{X}$ has variance-covariance matrix $\mathbf{\Sigma}$ and $\mathbf{A}$ is an $m \times p$ matrix of constants, prove that the variance-covariance matrix of $\mathbf{AX}$ is $\mathbf{A\Sigma A}'$. Start with the definition of a variance-covariance matrix:

   $$V(\mathbf{Z}) = E(\mathbf{Z} - \boldsymbol{\mu}_z)(\mathbf{Z} - \boldsymbol{\mu}_z)'.$$

5. If the $p \times 1$ random vector $\mathbf{X}$ has mean $\boldsymbol{\mu}$ and variance-covariance matrix $\mathbf{\Sigma}$, show $\mathbf{\Sigma} = E(\mathbf{XX}') - \boldsymbol{\mu}\boldsymbol{\mu}'$.

6. Let the $p \times 1$ random vector $\mathbf{X}$ have mean $\boldsymbol{\mu}$ and variance-covariance matrix $\mathbf{\Sigma}$, and let $\mathbf{c}$ be a $p \times 1$ vector of constants. Find $V(\mathbf{X}+\mathbf{c})$. Show your work. This is important because it tells us we can always pretend the mean equals zero when calculating covariance matrices.

7. Let $\mathbf{X}$ be a $p \times 1$ random vector with mean $\boldsymbol{\mu}_x$ and variance-covariance matrix $\boldsymbol{\Sigma}_x$, and let $\mathbf{Y}$ be an $r \times 1$ random vector with mean $\boldsymbol{\mu}_y$ and variance-covariance matrix $\boldsymbol{\Sigma}_y$. Define $C(\mathbf{X}, \mathbf{Y})$ by the $p \times r$ matrix $C(\mathbf{X}, \mathbf{Y}) = E\left((\mathbf{X} - \boldsymbol{\mu}_x)(\mathbf{Y} - \boldsymbol{\mu}_y)'\right)$.

   (a) What is the (i, j) element of $C(\mathbf{X}, \mathbf{Y})$?

   (b) Find an expression for $V(\mathbf{X}+\mathbf{Y})$ in terms of $\boldsymbol{\Sigma}_x$, $\boldsymbol{\Sigma}_y$ and $C(\mathbf{X}, \mathbf{Y})$. Show your work.

   (c) Simplify further for the special case where $Cov(X_i, Y_j) = 0$ for all i and j.

   (d) Let $\mathbf{c}$ be a $p \times 1$ vector of constants and $\mathbf{d}$ be an $r \times 1$ vector of constants. Find $C(\mathbf{X} + \mathbf{c}, \mathbf{Y} + \mathbf{d})$. Show your work.

8. Let X_1 be Normal(μ_1, σ_1^2), and X_2 be Normal(μ_2, σ_2^2), independent of X_1. What is the joint distribution of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$? What is required for Y_1 and Y_2 to be independent?

9. Let $\mathbf{X} = (X_1, X_2, X_3)'$ be multivariate normal with

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Let $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$. Find the joint distribution of Y_1 and Y_2.

10. Let $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{X}$ is an $n \times p$ matrix of known constants, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown constants, and $\boldsymbol{\epsilon}$ is multivariate normal with mean zero and covariance matrix $\sigma^2 \mathbf{I}_n$, where $\sigma^2 > 0$ is a constant. In the following, you may use $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$ without proof.

   (a) What is the distribution of $\mathbf{Y}$?

   (b) The maximum likelihood estimate (MLE) of $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$. What is the distribution of $\hat{\boldsymbol{\beta}}$? Show the calculations.

   (c) Let $\widehat{\mathbf{Y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$. What is the distribution of $\widehat{\mathbf{Y}}$? Show the calculations.

   (d) Let the vector of residuals $\mathbf{e} = (\mathbf{Y} - \widehat{\mathbf{Y}})$. What is the distribution of $\mathbf{e}$? Show the calculations. Simplify both the expected value (which is zero) and the covariance matrix.