STA 431s09 Assignment 1

Do this review assignment in preparation for the quiz on Wednesday, Jan. 14th. The problems are practice for the quiz, and are not to be handed in.

In problems 1 through 18, matrices of constants (like $\mathbf{A}$) are in boldface, and scalars (like a, b, c) are lower case in italics. Assume that they are of the right dimensions for the indicated operations, unless otherwise noted.

1. Which statement is true?
   (a) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
   (b) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{BA}+\mathbf{CA}$
   (c) Both a and b
   (d) Neither a nor b
2. Which statement is true?
   (a) $a(\mathbf{B}+\mathbf{C}) = a\mathbf{B}+a\mathbf{C}$
   (b) $a(\mathbf{B}+\mathbf{C}) = \mathbf{B}a+\mathbf{C}a$
   (c) Both a and b
   (d) Neither a nor b
3. Which statement is true?
   (a) $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{AB}+\mathbf{AC}$
   (b) $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$
   (c) Both a and b
   (d) Neither a nor b
4. Let $\mathbf{A}'$ denote $\mathbf{A}$ transpose. Which statement is true?
   (a) $(\mathbf{AB})' = \mathbf{A}'\mathbf{B}'$
   (b) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
   (c) Both a and b
   (d) Neither a nor b
5. Which statement is true?
   (a) $\mathbf{A}'' = \mathbf{A}$
   (b) $\mathbf{A}''' = \mathbf{A}'$
   (c) Both a and b
   (d) Neither a nor b

6. Suppose that the square matrices $\mathbf{A}$ and $\mathbf{B}$ both have inverses. Which statement is true?
   (a) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$
   (b) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
   (c) Both a and b
   (d) Neither a nor b
7. Which statement is true?
   (a) $(\mathbf{A}+\mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
   (b) $(\mathbf{A}+\mathbf{B})' = \mathbf{B}' + \mathbf{A}'$
   (c) $(\mathbf{A}+\mathbf{B})' = (\mathbf{B}+\mathbf{A})'$
   (d) All of the above
   (e) None of the above
8. The *trace* of a square matrix is the sum of its diagonal elements; it's denoted $tr(\mathbf{A})$. Which statement is true?
   (a) $tr(\mathbf{A}+\mathbf{B}) = tr(\mathbf{A}) + tr(\mathbf{B})$
   (b) $tr(\mathbf{A}+\mathbf{B}) = tr(\mathbf{B}) + tr(\mathbf{A})$
   (c) Both a and b
   (d) Neither a nor b
9. Which statement is true?
   (a) $a\,tr(\mathbf{B}) = tr(a\mathbf{B})$.
   (b) $tr(\mathbf{B})a = tr(a\mathbf{B})$
   (c) Both a and b
   (d) Neither a nor b
10. Which statement is true?
   (a) $(a+b)\mathbf{C} = a\mathbf{C} + b\mathbf{C}$
   (b) $(a+b)\mathbf{C} = \mathbf{C}a + \mathbf{C}b$
   (c) $(a+b)\mathbf{C} = \mathbf{C}(a+b)$
   (d) All of the above
   (e) None of the above

11. Recall that $\mathbf{A}$ symmetric means $\mathbf{A} = \mathbf{A}'$. Let $\mathbf{X}$ be an n by p matrix. Prove that $\mathbf{X}'\mathbf{X}$ is symmetric.

12. Recall that an inverse of the matrix $\mathbf{A}$ (denoted $\mathbf{A}^{-1}$) is defined by two properties: $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Prove that inverses are unique, as follows. Let $\mathbf{B}$ and $\mathbf{C}$ both be inverses of $\mathbf{A}$. Show that $\mathbf{B} = \mathbf{C}$.

13. Let $\mathbf{X}$ be an n by p matrix with $n \neq p$. Why is it incorrect to say that $(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{X}^{-1}\mathbf{X}'^{-1}$?

14. Suppose that the square matrices $\mathbf{A}$ and $\mathbf{B}$ both have inverses. Prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$. You have two things to show.

15. Let $\mathbf{A}$ be a square matrix with the determinant of $\mathbf{A}$ (denoted $|\mathbf{A}|$) equal to zero. What does this tell you about $\mathbf{A}^{-1}$?

16. Let $\mathbf{A}$ be a square matrix, and $\mathbf{A}^{-1}$ exists. Show that $(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$.

17. Let $\mathbf{A}$ be a square symmetric matrix, and $\mathbf{A}^{-1}$ exists. Show that $\mathbf{A}^{-1}$ is also symmetric.

18. Let $\mathbf{a}$ be an $n \times 1$ matrix of constants. How do you know $\mathbf{a}'\mathbf{a} \geq 0$?

19. Let the random vector $\mathbf{X} = (X_1, \ldots, X_n)'$ have density $f(\mathbf{x})$. For a general function g, use the rule $E[g(\mathbf{X})] = \int \cdots \int g(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}$ as if it were a definition. Prove

    (a) $E[a] = a$, where a is a constant.

    (b) $E[aX] = aE[X]$, where a is a constant.

    (c) $E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i]$

    (d) If X and Y are independent, $E[XY] = E[X]E[Y]$. Draw an arrow to the place in your answer where you use independence, and write "This is where I use independence."

    In all the remaining questions about variances and covariances, you should just use the linear properties of expectation that you have proved in Question 19, and avoid using integrals.

20. Denoting $E[X]$ by μ_x, define the variance $Var(X) = E[(X - \mu_x)^2]$. Show that $Var(X) = E[X^2] - \mu_x^2$.

21. Define the covariance of X and Y by $Cov[X, Y] = E[(X - \mu_x)(Y - \mu_y)]$. Show that $Cov[X, Y] = E[XY] - \mu_x\mu_y$.

22. In the following, X and Y are random variables, while a and b are fixed constants. For each pair of statements below, one is true and one is false (that is, not true in general). State which one is true, and prove it. Zero marks if you prove both statements are true, even if one of the proofs is correct.

   (a) $Var(aX) = aVar(X)$ or $Var(aX) = a^2Var(X)$
   (b) $Var(aX + b) = a^2Var(X) + b^2$ or $Var(aX + b) = a^2Var(X)$
   (c) $Var(a) = 0$ or $Var(a) = a^2$
   (d) $Cov(X + a, Y + b) = Cov(X, Y) + ab$ or $Cov(X + a, Y + b) = Cov(X, Y)$
   (e) $Var(aX+bY) = a^2Var(X)+b^2Var(Y)$ or $Var(aX+bY) = a^2Var(X)+b^2Var(Y)+2abCov(X, Y)$

23. For each of the following distributions, derive a general expression for the Maximum Likelihood Estimator (MLE). You don't have to do the second derivative test. Then use the data to calculate a numerical estimate.

   (a) $p(x) = \theta(1 - \theta)^x$ for $x = 0, 1, \ldots$, where $0 < \theta < 1$. Data: `4, 0, 1, 0, 1, 3, 2, 16, 3, 0, 4, 3, 6, 16, 0, 0, 1, 1, 6, 10`. Answer: 0.2061856
   (b) $f(x) = \frac{\alpha}{x^{\alpha+1}}$ for $x > 1$, where $\alpha > 0$. Data: `1.37, 2.89, 1.52, 1.77, 1.04, 2.71, 1.19, 1.13, 15.66, 1.43` Answer: 1.469102
   (c) $f(x) = \frac{\tau}{\sqrt{2\pi}}e^{-\frac{\tau^2x^2}{2}}$, for x real, where $\tau > 0$. Data: `1.45, 0.47, -3.33, 0.82, -1.59, -0.37, -1.56, -0.20` Answer: 0.6451059
   (d) $f(x) = \frac{1}{\theta}e^{-x/\theta}$ for $x > 0$, where $\theta > 0$. Data: `0.28, 1.72, 0.08, 1.22, 1.86, 0.62, 2.44, 2.48, 2.96` Answer: 1.517778