STA 413F2011 Assignment 9

Please read pages 319-325 in Section 6.2; Example 6.2.4 may be familiar, but note the additional details.

For this assignment, let $X_1, \ldots, X_n$ be a random sample from a *discrete* distribution with probability mass function $p(x;\theta), \theta \in \Omega$. The probability mass function posesses derivaties of all orders at the true parameter value θ_0. The support of the distribution is a finite set A, so that

$$\sum_{x \in A} p(x;\theta) = 1.$$

The set A does not depend on θ. In some of the questions below, X (without a subscript) is a random variable from this distribution.

1. Does the Binomial distribution fit the description of $p(x;\theta)$? How about the Poisson?
2. Answer True or False and justify your answer:
$$\frac{\partial^2}{\partial\theta^2} \sum_{x \in A} p(x;\theta) = \sum_{x \in A} \frac{\partial^2}{\partial\theta^2} p(x;\theta).$$
3. Let the *score* random variable $S = \frac{\partial}{\partial\theta} \ln p(X;\theta)$. Show $E(S) = 0$.
4. Show $Var(S) = I(\theta)$, where
$$I(\theta) = -E\left(\frac{\partial^2}{\partial\theta^2} \ln p(X;\theta)\right).$$
5. Let $X_1, \ldots, X_n$ be a random sample from $p(x;\theta)$, let $Y = u(X_1, \ldots, X_n)$ have expected value $E(Y) = k(\theta)$, and let $Z = \sum_{i=1}^n S_i$.
   (a) Prove $k'(\theta) = E(YZ)$.
   (b) What is $E(Z)$?
   (c) What is $Cov(Y, Z)$?
   (d) What is $Corr(Y, Z)$?
   (e) Using the fact that the absolute value of a correlation cannot be greater than one (proved in lecture but not requested here), establish the Cramér-Rao inequality of Theorem 6.2.1.
   (f) Give a lower bound for the variance of any unbiased estimator of θ, based on a random sample from this distribution.
6. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with parameter λ. Find the MLE of λ. Is it efficient? Answer Yes or No and prove your answer.
7. Let $X_1, \ldots, X_n$ be a random sample from a Binomial distribution with parameters 4 and θ. Find the MLE of θ. Is it efficient? Answer Yes or No and prove your answer.
8. Do Exercises 6.2.1, 6.2.2, 6.2.7 and 6.2.8.