STA 413F2011 Assignment 8

1. This question is based on material that you probably know already. However, the notation used in Statistics can be an obstacle for some students, so we will review the following basic rules.

   - The distributive law: $a(b+c) = ab + ac$. You may see this in a form like
   $$\theta \sum_{i=1}^n x_i = \sum_{i=1}^n \theta x_i$$
   - Power of a product is the product of powers: $(ab)^c = a^c\, b^c$. You may see this in a form like
   $$\left(\prod_{i=1}^n x_i \right)^\alpha = \prod_{i=1}^n x_i^\alpha$$
   - Multiplication is addition of exponents: $a^b a^c = a^{b+c}$. You may see this in a form like
   $$\prod_{i=1}^n \theta e^{-\theta x_i} = \theta^n \exp(-\theta \sum_{i=1}^n x_i)$$
   - Powering is multiplication of exponents: $(a^b)^c = a^{bc}$. You may see this in a form like
   $$(e^{\mu t + \frac{1}{2}\sigma^2 t^2})^n = e^{n\mu t + \frac{1}{2} n \sigma^2 t^2}$$
   - Log of a product is sum of logs: $\ln(ab) = \ln(a) + \ln(b)$. You may see this in a form like
   $$\ln \prod_{i=1}^n x_i = \sum_{i=1}^n \ln x_i$$
   - Log of a power is the exponent times the log: $\ln(a^b) = b\,\ln(a)$. You may see this in a form like
   $$\ln(\theta^n) = n \ln \theta$$
   - The log is the inverse of the exponential function: $\ln(e^a) = a$. You may see this in a form like
   $$\ln \left(\theta^n \exp(-\theta \sum_{i=1}^n x_i) \right) = n \ln \theta - \theta \sum_{i=1}^n x_i$$

Choose the correct answer.

(a) $\prod_{i=1}^{n} e^{x_i} =$

i. $\exp(\prod_{i=1}^{n} x_i)$

ii. e^{nx_i}

iii. $\exp(\sum_{i=1}^{n} x_i)$

(b) $\prod_{i=1}^{n} \lambda e^{-\lambda x_i} =$

i. $\lambda e^{-\lambda^n x_i}$

ii. $\lambda^n e^{-\lambda n x_i}$

iii. $\lambda^n \exp(-\lambda \sum_{i=1}^{n} x_i)$

iv. $\lambda^n \exp(-n\lambda \sum_{i=1}^{n} x_i)$

v. $\lambda^n \exp(-\lambda^n \sum_{i=1}^{n} x_i)$

(c) $\prod_{i=1}^{n} a_i^b =$

i. na^b

ii. a^{nb}

iii. $(\prod_{i=1}^{n} a_i)^b$

(d) $\prod_{i=1}^{n} a^{b_i} =$

i. na^{b_i}

ii. a^{nb_i}

iii. $\sum_{i=1}^{n} a^{b_i}$

iv. $a^{\prod_{i=1}^{n} b_i}$

v. $a^{\sum_{i=1}^{n} b_i}$

(e) $\left(e^{\lambda(e^t-1)}\right)^n =$

i. $ne^{\lambda(e^t-1)}$

ii. $e^{n\lambda(e^t-1)}$

iii. $e^{\lambda(e^{nt}-1)}$

iv. $e^{n\lambda(e^t-n)}$

(f) $\left(\prod_{i=1}^{n} e^{-\lambda x_i}\right)^2 =$

i. $e^{-2n\lambda x_i}$

ii. $e^{-2\lambda \sum_{i=1}^{n} x_i}$

iii. $2e^{-\lambda \sum_{i=1}^{n} x_i}$

(g) True, or False?

i. $\sum_{i=1}^n \frac{1}{x_i} = \frac{1}{\sum_{i=1}^n x_i}$

ii. $\prod_{i=1}^n \frac{1}{x_i} = \frac{1}{\prod_{i=1}^n x_i}$

iii. $\frac{a}{b+c} = \frac{a}{b} + \frac{a}{c}$

iv. $\ln(a+b) = \ln(a) + \ln(b)$

v. $e^{a+b} = e^a + e^b$

vi. $e^{a+b} = e^a e^b$

vii. $e^{ab} = e^a e^b$

viii. $\prod_{i=1}^n (x_i + y_i) = \prod_{i=1}^n x_i + \prod_{i=1}^n y_i$

ix. $\ln(\prod_{i=1}^n a_i^b) = b\sum_{i=1}^n \ln(a_i)$

x. $\sum_{i=1}^n \prod_{j=1}^n a_j = n\prod_{j=1}^n a_j$

xi. $\sum_{i=1}^n \prod_{j=1}^n a_i = \sum_{i=1}^n a_i^n$

xii. $\sum_{i=1}^n \prod_{j=1}^n a_{i,j} = \prod_{j=1}^n \sum_{i=1}^n a_{i,j}$

(h) Simplify as much as possible.

i. $\ln\prod_{i=1}^n \theta^{x_i}(1-\theta)^{1-x_i}$

ii. $\ln\prod_{i=1}^n \binom{m}{x_i}\theta^x(1-\theta)^{m-x_i}$

iii. $\ln\prod_{i=1}^n \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$

iv. $\ln\prod_{i=1}^n \theta(1-\theta)^{x_i-1}$

v. $\ln\prod_{i=1}^n \frac{1}{\theta}e^{-x_i/\theta}$

vi. $\ln\prod_{i=1}^n \frac{1}{\beta^\alpha\Gamma(\alpha)}e^{-x_i/\beta}x_i^{\alpha-1}$

vii. $\ln\prod_{i=1}^n \frac{1}{2^{\nu/2}\Gamma(\nu/2)}e^{-x_i/2}x_i^{\nu/2-1}$

viii. $\ln\prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$

ix. $\prod_{i=1}^n \frac{1}{\beta-\alpha}I(\alpha \le x_i \le \beta)$ (Express in terms of the minimum and maximum y_1 and y_n.)

2. For each of the following distributions, derive a general expression for the Maximum Likelihood Estimator (MLE). Don't bother with the second derivative test. Then use the data to calculate a numerical estimate; you should bring a calculator to the quiz in case you have to do something like this.

(a) $p(x) = \theta(1-\theta)^x$ for $x = 0, 1, \ldots$, where $0 < \theta < 1$. Data: `4, 0, 1, 0, 1, 3, 2, 16, 3, 0, 4, 3, 6, 16, 0, 0, 1, 1, 6, 10`. Answer: 0.2061856

(b) $f(x) = \frac{\alpha}{x^{\alpha+1}}$ for $x > 1$, where $\alpha > 0$. Data: `1.37, 2.89, 1.52, 1.77, 1.04, 2.71, 1.19, 1.13, 15.66, 1.43` Answer: 1.469102

(c) $f(x) = \frac{\tau}{\sqrt{2\pi}}e^{-\frac{\tau^2 x^2}{2}}$, for x real, where $\tau > 0$. Data: `1.45, 0.47, -3.33, 0.82, -1.59, -0.37, -1.56, -0.20` Answer: 0.6451059

(d) $f(x) = \frac{1}{\theta}e^{-x/\theta}$ for $x > 0$, where $\theta > 0$. Data: `0.28, 1.72, 0.08, 1.22, 1.86, 0.62, 2.44, 2.48, 2.96` Answer: 1.517778

3. Let $X_1, \ldots, X_N$ be a random sample from a Poisson distribution with parameter λ.

   (a) Derive a general formula for the maximum likelihood estimator (MLE) of λ. Show your work, but don't bother with the second derivative test.

   (b) Find the MLE $\widehat{\lambda}$ based on the following data: `7 7 6 4 2 5 2 3 7 2`.

4. Let $X_1, \ldots, X_n \stackrel{i.i.d.}{\sim} N(\mu, \sigma^2)$.

   (a) Derive a general formula for the maximum likelihood estimator (MLE) of the pair μ, σ^2. Show your work. Don't bother with anything like a second derivative test.

   (b) Find the MLE $(\widehat{\mu}, \widehat{\sigma}^2)$ based on the following data: `1.37 7.34 7.41 3.66 9.35 6.34 6.73 10.06`.

5. You have seen this model for simple regression through the origin before.

$$Y_i = \beta x_i + \epsilon_i$$

   for $i = 1, \ldots, n$, where $x_1, \ldots, x_n$ are fixed, observable constants, $\epsilon_1, \ldots, \epsilon_n$ are a random sample from a *normal* distribution with expected value zero and variance σ^2, and β and σ^2 are unknown constants.

   (a) Find the Maximum Likelihood Estimator of the pair (β, σ^2). Show your work.

   (b) Compute the pair of MLEs for these data:

x	4	2	3	1	3	3	1	3
y	31.04	10.09	-8.13	-1.38	15.7	11.36	-4.00	12.22

6. Let $X_1, \ldots, X_n$ be a random sample from distribution that is Uniform on the interval from 0 to θ.

   (a) Find the maximum likelihood estimator $\widehat{\theta}$. Show your work, including a rough sketch of the likelihood function.

   (b) Compute the MLE for these data: `1.35 1.57 1.34 2.44 0.26 1.81 1.37 1.53`.

Bring a calculator to the quiz!