STA 413F2011 Assignment 4

Do this assignment in preparation for Quiz Four in tutorial on Friday Oct. 14th. The problems are practice for the quiz, and are not to be handed in. See Formula Sheet 3; any necessary material from the formula sheet will be supplied with the quiz. Please start by reading Section 4.3 in the text: Pages 207-213.

1. Do Exercise 4.3.1 two ways:
   (a) Use the definition of convergence in distribution. Hint: Do it separately for $x < \mu$ and $x > \mu$; standardize $\overline{X}_n$.
   (b) Use limiting moment-generating functions.

2. Let $f_{X_n}(x) = \frac{1}{4}I(x=0) + \frac{1}{2}I(x=1) + \frac{1}{4}I(x = \frac{n+1}{n})$.
   (a) Is X_n discrete, or is it continuous?
   (b) What is $F_{X_n}(x)$? Write it using indicators.
   (c) Find $g(x) = \lim_{n\to\infty} f_{X_n}(x)$. Consider the cases $x=0$, $x=1$ and x equals something else separately. Is $g(x)$ a probability distribution?
   (d) Find $G(x) = \lim_{n\to\infty} F_{X_n}(x)$. Your answer should apply to all x. Is $G(x)$ a cumulative distribution function?
   (e) Let X be a Bernoulli random variable with $\theta = \frac{3}{4}$. Denote the cumulative distribution function of X by $F_X(x)$. At what points is $F_X(x)$ discontinuous? Does $G(x)$ equal $F_X(x)$ except possibly at those points? Does this mean $X_n \stackrel{d}{\rightarrow} X$? (Check the definition.)
   (f) Do we have $\lim_{n\to\infty} E(X_n) = E(X)$? Answer Yes or No. Show your work.
   (g) Do we have $\lim_{n\to\infty} Var(X_n) = Var(X)$? Answer Yes or No. Show your work.

3. Let the discrete random variable X_n have probability mass function $p_{X_n}(x) = \frac{1}{3}I(x=0) + \frac{2}{3}\left(\frac{n-1}{n}\right)I(x=1) + \frac{2}{3n}I(x=n)$.
   (a) What is $F_{X_n}(x)$? You may write the answer as a case function, or you may write it using indicators.
   (b) Let $p(x) = \lim_{n\to\infty} p_{X_n}(x)$. Consider the cases $x=0$, $x=1$ and x equals something else separately. Is $p(x)$ a probability distribution?
   (c) Let $F(x) = \lim_{n\to\infty} F_{X_n}(x)$.
      i. What is $F(x)$? Your answer should apply to all real x. You may write the answer as a case function, or you may write it using indicators.
      ii. Is $F(x)$ a cumulative distribution function? Does it correspond to $p(x)$?
      iii. We seem to have $X_n \stackrel{d}{\rightarrow} X$, where X is Bernoulli again. What is the parameter θ?
   (d) Do we have $\lim_{n\to\infty} E(X_n) = E(X)$? Answer Yes or No. Show your work.
   (e) Do we have $\lim_{n\to\infty} Var(X_n) = Var(X)$? Answer Yes or No. Show your work.

4. Prove $\displaystyle\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$. Hint: Use L'Hôpital's rule.

5. Let $X_1, \ldots, X_n$ be a random sample from a distribution for which the moment-generating function $M(t)$ exists, with $E(X_i) = \mu$. Use moment-generating functions to show $\overline{X}_n \stackrel{d}{\rightarrow} Y$, where Y is a "degenerate" random variable with $Pr\{Y = \mu\} = 1$. Hint: Use L'Hôpital's rule.

6. Do Exercise 4.3.2. What is the support of Z_n? To do this problem, use the definition of convergence in distribution.

7. Let $X_1, \ldots, X_n$ be a random sample from a distribution with a density that is uniform on the interval from zero to θ, and let Y_n denote the maximum. We will investigate the limiting behaviour of $T_n = n(1 - \frac{Y_n}{\theta})$.

   (a) What is the support of Y_n?

   (b) What is the support of T_n?

   (c) If T_n has a limiting distribution, what does the support of that limiting distribution have to be?

   (d) Now write the cumulative distribution function of T_n as $Pr\{n(1 - \frac{Y_n}{\theta}) \leq t\}$, simplify, and take the limit as $n \rightarrow \infty$. Use Problem 4.

   (e) If you recognize the result, great; what is the limiting distribution called? otherwise, differentiate it to obtain a familiar density.

8. Do Exercise 4.3.3. Hints: Assume the cumulative distribution function F is strictly increasing and therefore has a unique inverse. What is the support of Z_n?

9. Do Exercise 4.3.5 using the definition .

10. Do Exercise 4.3.6. Hints: First find the distribution of Z_n without proof, then standardize. Recall that for any cumulative distribution function, $\lim_{x \rightarrow \infty} F(x) = 1$ and $\lim_{x \rightarrow -\infty} F(x) = 0$.

11. Do Exercise 4.3.7. Hint: Limiting moment-generating functions and Problem 4.

12. Do Exercise 4.3.8. Hint: Fearlessly apply L'Hôpital's rule.

13. Do Exercise 4.3.9. Hint: For this problem and the next one, it is helpful to recall that convergence in distribution refers to convergence of cumulative distribution functions, and not of the random variables themselves. Thus, one may freely replace a random variable with a more convenient one that has the same distribution. In this case, let $W_1, \ldots, W_n$ be a random sample from a Chi-square distribution with $\nu = 1$. Using moment-generating functions if necessary (this was a homework problem in Assignment One), verify that $\sum_{i=1}^{n} W_i$ has the same distribution as X. Now you can use the Central Limit Theorem.

14. Do Exercise 4.3.11. Hint: The sum of independent Poissons is $\ldots$?