# STA 413F2011 Assignment 1

Do this review assignment in preparation for Quiz One in tutorial on Friday Sept 16th. The problems are practice for the quiz, and are not to be handed in. Please start by reading Section 4.1 in the textbook. Also, you may as well print the formula sheet and use it as necessary to do the problems; a copy will be supplied with the quiz.

1. Let the random variable $X$ have a density that is uniform on the interval from zero to $\theta$.
   (a) Write the density of $X$ using indicator functions[1].
   (b) Write the cumulative distribution function of $X$ using indicator functions.
   (c) Sketch the cumulative distribution function of $X$. Make sure your picture shows what the function is like *for all real* $x$.
2. Let the discrete random variable $X$ have probability mass function
   $$p(x) = \frac{x}{6} I(x = 1, 2, 3).$$
   Sketch the cumulative distribution function of $X$. Make sure your picture shows what the function is like *for all real* $x$.
3. Find $E(X)$ for the distribution of Problem 2. Show your work.
4. Let $X$ have a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$. Find $E[X^k]$, where $k$ is a positive integer. Show your work.
5. Let $X_1, \ldots, X_n$ be a random sample from a distribution (not necessarily normal) with mean $\mu$ and variance $\sigma^2$. Find the expected value and variance of $\overline{X}$. Show your work.
6. Denote the moment-generating function of a random variable $Y$ by $M_Y(t)$. The moment-generating function is defined by $M_Y(t) = E(e^{Yt})$.
   (a) Let $a$ be a constant. Prove that $M_{aX}(t) = M_X(at)$.
   (b) Let $X_1, \ldots, X_n$ be independent random variables. Prove that
   $$M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t).$$
   For convenience, you may assume that $X_1, \ldots, X_n$ are all continuous.
7. Let $X$ have a Poisson distribution with parameter $\lambda > 0$. Derive the moment-generating function of $X$. Use the moment-generating function to find $E(X)$ and $E(X^2)$.

[1] The indicator function notation was covered in lecture. Here is a copy of the overhead. There is a link on the course home page in case the one in this document does not work.

8. Let $X_1, \ldots, X_n$ be a collection of independent and identically distributed (iid) Poisson random variables with common parameter $\lambda > 0$. Using moment-generating functions, find the distribution of $Y = \sum_{i=1}^n X_i$.

9. A Chisquare random variable with parameter $r > 0$ is a Gamma with $\alpha = r/2$ and $\beta = 2$. Let $X_1, \ldots, X_n$ be a random sample (that is, a collection of independent and identically distributed random variables) from a Chisquare distribution with parameter $r > 0$. Find the distribution of $Y = \sum_{i=1}^n X_i$.

10. For any collection of numbers $X_1, \ldots, X_n$, prove this alternative formula for the sample variance:

$$S^2 = \frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n-1} = \frac{1}{n-1}\left(\sum_{i=1}^n X_i^2 - n\overline{X}^2\right)$$

Use it to show that for a random sample from a distribution with mean $\mu$ and variance $\sigma^2$, the sample variance is an unbiased estimator of $\sigma^2$.

11. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with density $f(x)$ and distribution function $F(x)$.

    (a) Derive the distribution function of $Y_n = \max(X_1, \ldots, X_n)$.

    (b) Derive the density of $Y_n$.

    (c) Derive the distribution function of $Y_1 = \min(X_1, \ldots, X_n)$.

    (d) Derive the density of $Y_1$.

12. Let $X_1, \ldots, X_n$ be a random sample from a uniform density from zero to $\theta$.

    (a) Find the expected value of this distribution. Show your work.

    (b) Prove that the statistic $T_1 = 2\overline{X}$ is unbiased for $\theta$.

    (c) Prove that the statistic $T_2 = X_1 + X_2$ is unbiased for $\theta$.

    (d) Find an unbiased estimator for $\theta$ based on the sample maximum. Of course you will use your answer to Question 11.

13. Please do problems 4.1.4, 4.1.6 and 4.1.7 from the text.