STA 413F2008 Assignment 6

Please take a look at Section 4.3.2. We proved the delta method another way in lecture. Also, please read pages 224-225. The following questions are practice for Test 3 and the Final Examination. They are not to be handed in.

1. Let $T_n \stackrel{P}{\rightarrow} \theta$, and let Y_n that is between T_n and θ. Prove that $Y_n \stackrel{P}{\rightarrow} \theta$. We will call this the Squeeze Theorem for Convergence in Probability.

2. The various parts of this question will lead you through the proof of the Delta Method for the special but important case of a function of the sample mean. That is, let $X_1, \ldots, X_n$ be a random sample from a distribution with expected value μ and variance σ^2, and let $g(x)$ be a function with $g'(\mu) \neq 0$ and $g''(x)$ continuous at $x = \mu$. Then

$$\sqrt{n}(g(\overline{X}_n) - g(\mu)) \stackrel{d}{\rightarrow} Y \sim N(0, g'(\mu)^2\sigma^2)$$

   (a) First, use Taylor's Theorem (on the formula sheet) to re-write

$$\sqrt{n}(g(\overline{X}_n) - g(\mu)), \tag{1}$$

   distributing $\sqrt{n}$. You now have two terms.

   (b) How do you know $\sqrt{n}(\overline{X}_n - \mu)$ converges in probability to a normal random variable? Cite something from the formula sheet.

   (c) How do you know $g''(\mu^*) \stackrel{P}{\rightarrow} g''(\mu)$? Use the formula sheet as well as another problem from this assignment.

   (d) Now show that the second of the two terms from Problem 2a converges in probability to zero. When you use something from the formula sheet, cite it.

   (e) Now show that the first term from Problem 2a converges in distribution to something. What is its target distribution? Again, when you use something from the formula sheet, cite it.

   (f) Now apply Slutsky (c) for convergence in distribution.

   (g) Now do the last step. You are using a particular continuous function $g(a, b)$. Write it in terms of a and b.

3. Do Exercises 4.4.11 and 4.4.12.

4. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter θ. Find the limiting distribution of

$$Z_n = 2\sqrt{n}\left(\sin^{-1}\sqrt{\overline{X}_n} - \sin^{-1}\sqrt{\theta}\right).$$

   Hint: $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$.

5. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with parameter θ.

   (a) Find a variance-stabilizing transformation. That is, find a function $g(x)$ such that the limiting distribution of

   $$Y_n = \sqrt{n}[g(\overline{X}_n) - g(\theta)]$$

   does not depend on θ.

   (b) To check, find the limiting distribution of Y_n.

6. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on $(0, \theta)$.

   (a) Find a variance-stabilizing transformation. That is, find a function $g(x)$ such that the limiting distribution of

   $$Y_n = \sqrt{n}[g(2\overline{X}_n) - g(\theta)]$$

   does not depend on θ.

   (b) To check, find the limiting distribution of Y_n.

7. Let $X_1, \ldots, X_n$ be a random sample from a chi-square distribution with parameter ν.

   (a) Find a variance-stabilizing transformation. That is, find a function $g(x)$ such that the limiting distribution of

   $$Y_n = \sqrt{n}[g(\overline{X}_n) - g(\nu)]$$

   does not depend on ν.

   (b) To check, find the limiting distribution of Y_n.

8. Let $X_1, \ldots, X_n$ be a random sample from a binomial distribution with parameters m and θ. The constant m is fixed and known.

   (a) Find a variance-stabilizing transformation. That is, find a function $g(x)$ such that the limiting distribution of

   $$Y_n = \sqrt{n}[g(\overline{X}_n/m) - g(\theta)]$$

   does not depend on θ.

   (b) To check, find the limiting distribution of Y_n.