STA 413F2008 Assignment 2

There will be several assignments for Test 2; this is the first one. The problems are practice for the test, and are not to be handed in. Please start by reading Section 4.2 in the textbook. You will use the theorems, but you are not responsible for any of the proofs unless they appear as homework problems. Pay attention to Theorem 4.2.4, which is sometimes called the Continuous Mapping Theorem; it is useful, and an even more useful version will be given later in lecture.

1. Do problem 4.2.1, except just prove $a_n \to a$ implies $a_n \stackrel{P}{\to} a$. My solution uses Exercise 16.

2. Let $T_1, T_2, \ldots$ be a sequence of real-valued random variables. If $\lim_{n\to\infty} E(T_n) = \theta$ and $\lim_{n\to\infty} Var(T_n) = 0$, show that $T_n \stackrel{P}{\to} \theta$. We will call this the *Variance Rule*.

3. Let $P(T_n = 0) = \frac{n-1}{n}$ and $P(T_n = n) = \frac{1}{n}$.

   (a) Show $T_n \stackrel{P}{\to} 0$.

   (b) Does $E(T_n) \to 0$?

   (c) Does $Var(T_n) \to 0$?

   This shows you that the Variance Rule is a sufficient but not a necessary condition for convergence in probability.

4. Do Problem 4.2.2.

5. Do Problem 4.2.3. You have already shown something that is more helpful than the hint.

6. Do Problem 4.2.4.

7. Do Problem 4.2.5.

8. Let $X_1, \ldots, X_n$ be a random sample from a distribution with expected value μ and variance σ^2. The Law of Large Numbers says $\overline{X}_n \stackrel{P}{\to} \mu$. Show that $\overline{X}_n + a \stackrel{P}{\to} \mu + a$, where a is a constant. A theorem in the text makes this quite easy.

9. As before, let $X_1, \ldots, X_n$ be a random sample from a distribution with expected value μ and variance σ^2. Prove that $3\overline{X}_n \stackrel{P}{\to} 3\mu$.

10. Let $X_1, \ldots, X_n$ be a random sample from a distribution with expected value μ and variance σ_x^2. Independently of $X_1, \ldots, X_n$, let $Y_1, \ldots, Y_n$ be a random sample from a distribution with the same expected value μ and variance σ_y^2. Let Let $T_n = \alpha\overline{X}_n + (1-\alpha)\overline{Y}_n$, where α is a constant between zero and one.

    (a) How do you know $\overline{X}_n$ and $\overline{Y}_n$ are independent?

    (b) Is T_n an unbiased estimator of μ? Answer Yes or No and show your work.

    (c) Is T_n a consistent estimator of μ? Answer Yes or No and show your work.

    (d) What value of α would make the estimator T_n most accurate in terms of having the smallest possible variance? Show your work.

11. Let $X_1, \ldots, X_n$ be independent random variables with a continuous uniform distribution on $[0, \theta]$, and let $Y_n = \max(X_1, \ldots, X_n)$.

(a) Using the definition of convergence in probability, show that Y_n is a consistent estimator of θ.

(b) Show that $2\overline{X}_n$ is also consistent for θ.

12. Let X be a random variable with expected value μ and variance σ^2. Show that $\frac{X}{n} \stackrel{P}{\rightarrow} 0$.

13. Now drop the assumption that X has a mean and variance; for example X could be Cauchy. Again, show that $\frac{X}{n} \stackrel{P}{\rightarrow} 0$.

14. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with parameter θ, and let $T_n = nY_1$, where $Y_1 = \min(X_1, \ldots, X_n)$.

    (a) Find the probability density function of T_n. Don't forget the support.

    (b) Is T_n unbiased? Answer Yes or No. You can use the formula sheet.

    (c) Is T_n consistent? Justify your answer by giving an explicit formula for $P(|T_n - \theta| < \epsilon)$.

15. A model for simple regression through the origin is

$$Y_i = \beta x_i + \epsilon_i$$

for $i = 1, \ldots, n$, where $\epsilon_1, \ldots, \epsilon_n$ are a random sample from a distribution with expected value zero and variance σ^2, and β and σ^2 are unknown constants.

    (a) What is $E(Y_i)$?

    (b) What is $Var(Y_i)$?

    (c) Find the Least Squares estimate of β by minimizing $Q = \sum_{i=1}^n (Y_i - \beta x_i)^2$ over all values of β. Let $\widehat{\beta}_n$ denote the point at which Q is minimal.

    (d) Is $\widehat{\beta}_n$ unbiased? Answer Yes or No and show your work.

    (e) Give a sufficient condition for $\widehat{\beta}_n$ to be consistent. Show your work. Remember, in this model the x_i are fixed constants, not random variables.

    (f) Let $\widehat{\beta}_{2,n} = \frac{\overline{Y}_n}{\overline{x}_n}$. Is $\widehat{\beta}_{2,n}$ unbiased? Consistent? Answer Yes or No to each question and show your work.

    (g) Prove that $\widehat{\beta}_n$ is a more accurate estimator than $\widehat{\beta}_{2,n}$ in the sense that it has smaller variance. Hint: The sample variance of the independent variable values cannot be negative.

16. Let $T_1, T_2, \ldots$ be a sequence of random variables. Let T be another random variable, and let θ be a constant.

    (a) Prove $T_n \stackrel{a.s.}{\rightarrow} T \Rightarrow T_n \stackrel{P}{\rightarrow} T$. Recall the approach from lecture. $T_n \stackrel{a.s.}{\rightarrow} T$ means there is a set A contained in the sample space $\mathcal{C}$ with $\lim_{n\rightarrow\infty} T_n(c) = T(c)$ for every $c \in A$. Let $\epsilon > 0$ be given, and define $A_n = \{c \in A : |T_k(c) - T(c)| < \epsilon \text{ for all } k \geq n\}$, and so on.

    (b) Prove $T_n \stackrel{P}{\rightarrow} \theta \Rightarrow T_n \stackrel{d}{\rightarrow} \theta$. The trick is to make sure ϵ is small enough so that the point you are considering is outside the interval from $\theta - \epsilon$ to $\theta + \epsilon$.

    (c) Prove $T_n \stackrel{d}{\rightarrow} \theta \Rightarrow T_n \stackrel{P}{\rightarrow} \theta$. This holds only when the target is a constant, not a (non-degenerate) random variable.