STA 413F2008 Assignment 1

Do this review assignment in preparation for Test One on Wednesday Sept 24th. The problems are practice for the quiz, and are not to be handed in. Please start by reading Section 4.1 in the textbook. Also, you may as well print the formula sheet and use it as necessary to do the problems; a copy will be supplied with the test.

1. Let the random variable X have a density that is uniform on the interval from zero to θ.
   (a) Write the density of X using indicator functions.
   (b) Write the cumulative distribution function of X using indicator functions.
   (c) Sketch the cumulative distribution function of X. Make sure your picture shows what the function is like *for all real* x.
   (d) Derive the moment-generating function of X.
2. Let the discrete random variable X have probability mass function $p(x) = \frac{x}{6} I(x = 1, 2, 3)$. Sketch the cumulative distribution function of X. Make sure your picture shows what the function is like *for all real* x.
3. Let X have a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$. Find $E[X^k]$, where $k \geq 0$. Show your work.
4. Let X have a Poisson distribution with parameter $\lambda > 0$. Derive the moment-generating function of X.
5. Let $X_1, \ldots, X_n$ be a collection of independent and identically distributed (iid) Poisson random variables with common parameter $\lambda > 0$. Using moment-generating functions, find the distribution of $Y = \sum_{i=1}^n X_i$.
6. A Chisquare random variable with parameter $r > 0$ is a Gamma with $\alpha = r/2$ and $\beta = 2$. Let $X_1, \ldots, X_n$ be a random sample (that is, a collection of iid random variables) from a Chisquare distribution with parameter $r > 0$. Find the distribution of $Y = \sum_{i=1}^n X_i$.
7. Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution with density $f(x)$ and distribution function $F(x)$.
   (a) Derive the distribution function of $Y_n = \max(X_1, \ldots, X_n)$.
   (b) Derive the density of Y_n.
   (c) Derive the distribution function of $Y_1 = \min(X_1, \ldots, X_n)$.
   (d) Derive the density of Y_1.
8. Let $X_1, \ldots, X_n$ be a random sample from a distribution (not necessarily normal) with mean μ and variance σ^2. Find the expected value and variance of $\overline{X}$. Show your work.
9. Based on lecture notes (or on the text if you want to look in the index),
   (a) State and prove Markov's inequality for a continuous random variable X.
   (b) Use Markov's inequality to prove Chebyshev's inequality.
   (c) Use Chebyshev's inequality to prove the Weak Law of Large Numbers.

10. For any collection of numbers $X_1, \ldots, X_n$, prove this alternative formula for the sample variance:

$$\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1} = \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right)$$

11. Let $X_1, \ldots, X_n$ be a random sample from a distribution (not necessarily normal) with mean μ and variance σ^2. Prove that the sample variance S^2 is unbiased for σ^2.

12. Let $X_1, \ldots, X_n$ be a random sample from a uniform density from zero to θ.

    (a) Find the expected value of this distribution. Show your work.

    (b) Prove that the statistic $T_1 = 2\overline{X}$ is unbiased for θ.

    (c) Prove that the statistic $T_2 = X_1 + X_2$ is unbiased for θ.

    (d) Find an unbiased estimator for θ based on the sample maximum. Of course you will use your answer to Question 7.

13. Please do the following problems from the text: 4.1.4, 4.1.6, 4.1.7, 4.1.9, 4.1.19 (Answer is 275), 4.1.25, 4,1.26 and 4.1.27. Problems 4.1.25 and 4.1.26 refer to material earlier in the textbook. For your convenience, the relevant pages are reproduced at the end of this assignment.

cannot be improved without further assumptions about the distribution of X. ∎

Definition 1.10.1.

A function ϕ defined on an interval (a, b), $-\infty \leq a < b \leq \infty$, is said to be a **convex** function if for all x, y in (a, b) and for all $0 < \gamma < 1$,

$$\phi[\gamma x + (1 - \gamma)y] \leq \gamma\phi(x) + (1 - \gamma)\phi(y). \tag{1.10.4}$$

We say ϕ is **strictly convex** if the above inequality is strict.

Depending on existence of first or second derivatives of ϕ, the following theorem can be proved.

Theorem 1.10.4. *If ϕ is differentiable on (a, b) then*

(a) ϕ is convex if and only if $\phi'(x) \leq \phi'(y)$, for all $a < x < y < b$,

(b) ϕ is strictly convex if and only if $\phi'(x) < \phi'(y)$, for all $a < x < y < b$.

If ϕ is twice differentiable on (a, b) then

(a) ϕ is convex if and only if $\phi''(x) \geq 0$, for all $a < x < b$,

(b) ϕ is strictly convex if $\phi''(x) > 0$, for all $a < x < b$.

Of course the second part of this theorem follows immediately from the first part. While the first part appeals to one's intuition, the proof of it can be found in most analysis books; see, for instance, Hewitt and Stromberg (1965). A very useful probability inequality follows from convexity.

Theorem 1.10.5 (Jensen's Inequality). *If ϕ is convex on an open interval I and X is a random variable whose support is contained in I and has finite expectation, then*

$$\phi[E(X)] \leq E[\phi(X)]. \tag{1.10.5}$$

If ϕ is strictly convex then the inequality is strict, unless X is a constant random variable.

Theorem 3.3.1. *Let X have a $\chi^2(r)$ distribution. If $k > -r/2$ then $E(X^k)$ exists and it is given by*

$$E(X^k) = \frac{2^k \Gamma\left(\frac{r}{2}+k\right)}{\Gamma\left(\frac{r}{2}\right)}, \quad \text{if } k > -r/2. \tag{3.3.4}$$

Proof. Note that

$$E(X^k) = \int_0^\infty \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{r/2}} x^{(r/2)+k-1} e^{-x/2}\, dx.$$

Make the change of variable $u = x/2$ in the above integral. This results in

$$E(X^k) = \int_0^\infty \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{(r/2)-1}} 2^{(r/2)+k-1} u^{(r/2)+k-1} e^{-u}\, du.$$

This yields the desired result provided that $k > -(r/2)$. ■

Notice that if k is a nonnegative integer then $k > -(r/2)$ is always true. Hence, all moments of a χ^2 distribution exist and the kth moment is given by (3.3.4).

Example 3.3.5. Let X be $\chi^2(10)$. Then, by Table II of Appendix C, with $r = 10$,

$$\begin{aligned} P(3.25 \le X \le 20.5) &= P(X \le 20.5) - P(X \le 3.5) \\ &= 0.975 - 0.025 = 0.95. \end{aligned}$$

Again, as an example, if $P(a < X) = 0.05$, then $P(X \le a) = 0.95$, and thus $a = 18.3$ from Table II with $r = 10$. ■