STA 347F 2003 Makeup Test
Aids allowed: Calculator, but you won't need it.

1. Let $X_0, X_1, \ldots$ be a stationary Markov chain with transition matrix

	0	1	2
0	2/6	3/6	1/6
1	3/6	1/6	2/6
2	1/6	2/6	3/6

,

   and let $Pr\{X_0 = 0\} = Pr\{X_0 = 1\} = Pr\{X_0 = 2\} = \frac{1}{3}$. Notice that the matrix is *doubly stochastic*, and we are starting with the stationary distribution.

   (a) (8 Points) What is $Pr\{X_0 = 0, X_1 = 1, X_2 = 0\}$? Show some work.

   (b) (2 Points) What is $Pr\{X_1 = 2\}$?

   (c) (2 Points) What is $Pr\{X_3 = 2\}$?

   (d) (2 Points) What is $Pr\{X_n = 2\}$?

   (e) (4 Points) What is $\lim_{n\to\infty} Pr\{X_n = 2\}$?

   (f) (4 Points) What is $\lim_{n\to\infty} P_{02}^{(n)}$?

   (g) (10 Points) Starting in State 1, what is the probability of reaching State 0 before State 2? Show your work.

2. (12 Points) Let $X_0, X_1, \ldots$ be a regular stationary Markov chain with finite state space. What is $\lim_{n\to\infty} Pr\{X_0 = j | X_n = i\}$? If this quantity does not exist, just write "The limit does not exist." Otherwise, find it and show your calculations.

3. Consider a stationary Markov chain with transition probabilities $P_{i,j} = \alpha_j$, where $\alpha_j > 0$ and $\sum_{j=0}^{\infty} \alpha_j = 1$, for $i = 0, 1, \ldots$ and $j = 0, 1, \ldots$.

   (a) (2 Points) Is this Markov chain irreducible? Answer Yes or No.

   (b) (4 Points) What is the period of this Markov. How do you know?

   (c) (6 Points) What is $P_{i,j}^{(2)}$? Show your work.

   (d) (6 Points) What is $P_{i,j}^{(n)}$?

   (e) (6 Points) Show that State j is either transient or recurrent. Then write "Transient" or "Recurrent," and *circle your answer*.

   (f) (8 Points) Find the stationary distribution. Start with an expression for π_j that comes from the formula for matrix multiplication; it's easier than you might think.

4. (12 Points) Let $\{X(t) : t \geq 0\}$ be a Poisson process with rate λ, and let W_1 be the waiting time until the first event. Derive the density of W_1 given $X(t) = 1$. Be sure to indicate where the density is non-zero.

5. (12 Points) Let $\{X_1(t), X_2(t), \ldots, X_n(t)\}$ be *independent* Poisson processes with rates $\lambda_1, \ldots, \lambda_n$, respectively. What is the probability that at least r units of time pass before the first event from *any* process?

Total marks = 100 points

MUAnsw1

Jenny's answers to the Makeup Test

(1) (a) $Pr\{X_0=0, X_1=1, X_2=0\}$

$= Pr\{X_0=0\} Pr\{X_1=1 / X_0=0\} Pr\{X_2=0 / \cancel{X_0=0}, X_1=1\}$

$= Pr\{X_0=0\} P_{01} \; P_{10} = \frac{1}{\cancel{3}} \frac{\cancel{3}}{6} \frac{3}{6} = \frac{3}{3 \cdot 2 \cdot 6} = \frac{1}{12}$

(b) $\frac{1}{3}$ (Start with stationary distribution)

(c) $\frac{1}{3}$

(d) $\frac{1}{3}$

(e) $\frac{1}{3}$

(f) $\frac{1}{3}$

(g) Let $A = \{$Reach 0 before 2$\}$

$u = Pr\{A / X_0 = 1\}$

$= Pr\{A / \cancel{X_0=1}, X_1=0\} P_{10} + Pr\{A / \cancel{X_0=1}, X_1=1\} P_{11} + Pr\{A / \cancel{X_0=1}, X_1=2\} P_{02}$

$= 1 \cdot P_{10} + u P_{11} + 0 = \frac{3}{6} + \frac{1}{6} u$

$\Leftrightarrow 5/6 \, u = \frac{3}{6}$

$\Leftrightarrow u = \frac{3}{5}$

(2) $\lim_{n\to\infty} Pr\{X_0 = j \mid X_n = i\} = \lim_{n\to\infty} \dfrac{Pr\{X_0 = j, X_n = i\}}{Pr\{X_n = i\}}$

$= \lim_{n\to\infty} \dfrac{Pr\{X_n = i \mid X_0 = j\}\, Pr\{X_0 = j\}}{Pr\{X_n = i\}}$

$= Pr\{X_0 = j\} \dfrac{\lim_{n\to\infty} P_{ji}^{(n)}}{\lim_{n\to\infty} Pr\{X_n = i\}} = Pr\{X_0 = j\} \dfrac{\pi_i}{\pi_i}$

$= Pr\{X_0 = j\}$

(3) (a) Yes

(b) Period = 1 because $P_{jj} > 0$ all j

(c) $P_{ij}^{(2)} = \sum_{k=0}^{\infty} P_{ik} P_{kj} = \sum_{k=0}^{\infty} \alpha_k \alpha_j = \alpha_j \underbrace{\sum_{k=0}^{\infty} \alpha_k}_{=1} = \alpha_j$

(d) $P_{ij}^{(n)} = \alpha_j$ (Since $\underset{\sim}{P}^2 = \underset{\sim}{P} \Rightarrow \underset{\sim}{P}^n = \underset{\sim}{P}$)

(e) $\sum_{n=1}^{\infty} P_{jj}^{(n)} = \sum_{n=1}^{\infty} \alpha_j = \infty$, so $\boxed{\text{Recurrent}}$

(f) $\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj} = \sum_{k=0}^{\infty} \pi_k \alpha_j = \alpha_j \sum_{k=0}^{\infty} \pi_k$

$= \alpha_j$

(4) For $0 < w \le t$, $\frac{d}{dw} \Pr\{W_1 \le w \mid X(t) = 1\}$

$$= \frac{d}{dw} \Pr\{X(w) = 1 \mid X(t) = 1\} = \frac{d}{dw} \frac{\Pr\{X(w)=1, X(t)=1\}}{\Pr\{X(t)=1\}}$$

$$= \frac{d}{dw} \frac{\Pr\{X(w)=1\} \Pr\{X(t) - X(w) = 0\}}{\Pr\{X(t) = 1\}}$$

$$= \frac{d}{dw} \frac{\cancel{e^{-\lambda w}}\, \cancel{\lambda} w \, \cancel{e^{-\lambda(t-w)}}}{\cancel{e^{-\lambda t}}\, \cancel{\lambda} t} = \frac{d}{dw}\left(\frac{w}{t}\right) = \frac{1}{t},$$

so the density is $\frac{1}{t} 1\{0 < w \le t\}$

(5) Let T_i be waiting time until first event from process i.

$$\Pr\left\{\bigcap_{i=1}^{n} T_i > r\right\} \overset{\text{ind}}{=} \prod_{i=1}^{n} \Pr\{T_i > r\} = \prod_{i=1}^{n} e^{-\lambda_i r}$$

$$= e^{-r \sum_{i=1}^{n} \lambda_i}$$