P4.8. The key is that given $N(t)=n$, $W_1, \ldots W_n$ are the ordered values of $U_1 \ldots U_n$, iid uniforms on $(0,t]$.

$$E[Z(t)] = \sum_{n=0}^{\infty} E[Z(t) \mid N(t)=n] \Pr\{N(t)=n\}$$

$$= 0 + \sum_{n=1}^{\infty} E\Big[\sum_{k=1}^{N(t)} \Theta_k(t) \mid N(t)=n\Big] \Pr\{N(t)=n\}$$

$$= \sum_{n=1}^{\infty} E\Big[\sum_{k=1}^{n} \varepsilon_k e^{-\alpha(t-W_k)} 1\{t \geq W_k\} \mid N(t)=n\Big] \Pr\{N(t)=n\}$$

$$= \sum_{n=1}^{\infty} E\Big[\sum_{k=1}^{n} \varepsilon_k e^{-\alpha(t-U_k)} \underbrace{1\{t \geq U_k\}}_{\text{Always have } 0<U_k\leq t \text{ So it's one}}\Big] \Pr\{N=n\}$$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{n} E\big[\varepsilon_k e^{-\alpha(t-U_k)}\big] \Pr\{N=n\}$$

$$\overset{\text{iid}}{=} \sum_{n=1}^{\infty} \sum_{k=1}^{n} \underbrace{E[\varepsilon_k]}_{\text{They are iid. call it } \mu} E\big[e^{-\alpha(t-U_k)}\big] \Pr\{N=n\}$$

$$= \mu \sum_{n=1}^{\infty} \sum_{k=1}^{n} e^{-\alpha t} \underbrace{E\big[e^{\alpha U_k}\big]}$$ $$\Pr\{N=n\}$$

Because $U_1 \ldots U_n$ are iid, this is a single constant that does not ~~depend~~ depend on k. Call it $E(e^{\alpha U})$

$$= \mu E[e^{\alpha U}] e^{-\alpha t} \underbrace{\sum_{n=1}^{\infty} n \Pr\{N=n\}}_{=\lambda t}$$

Now find this

P 4.8 continued

$$E[e^{\alpha u}] = \int_0^t e^{\alpha u} \frac{1}{t}\, du$$

$$= \frac{1}{\alpha t} \int_0^t e^{\alpha u} \alpha\, du$$

Let $y = \alpha u$
$dy = \alpha\, du$

u	$y = \alpha u$
t	αt
0	0

$$= \frac{1}{\alpha t} \int_0^{\alpha t} e^{y}\, dy$$

$$= \frac{1}{\alpha t}\left(e^{\alpha t} - 1\right), \text{ so}$$

$$E[Z(t)] = \mu e^{-\alpha t} \frac{1}{\alpha t}\left(e^{\alpha t} - 1\right)$$

$$= \boxed{\frac{\mu}{\alpha t}\left(1 - e^{-\alpha t}\right)}$$