P4.7.

$$h_i = E\left[\sum_{n=0}^{\infty} \beta^n c(X_n) \mid X_0 = i\right]$$

$$= E\left[E\left[\sum_{n=0}^{\infty} \beta^n c(X_n) \mid X_0 = i, X_1\right]\right]$$

$$= \sum_{j=0}^{\infty} E\left[\sum_{n=0}^{\infty} \beta^n c(X_n) \mid X_0 = i, X_1 = j\right] P\{X_1 = j \mid X_0 = i\}$$

MISSING THEOREM

$$\overset{\downarrow}{=} \sum_{j=0}^{\infty} E\left[\beta^0 c(i) + \sum_{n=1}^{\infty} \beta^n c(X_n) \mid X_1 = j\right] P_{ij}$$

$$= \sum_{j=0}^{\infty} \left(c(i) P_{ij} + E\left[\sum_{n=1}^{\infty} \beta^n c(X_n) \mid X_1 = j\right] P_{ij}\right)$$

$$= c(i) \underbrace{\sum_{j=0}^{\infty} P_{ij}}_{=1} + \beta \sum_{j=0}^{\infty} E\left[\sum_{n=1}^{\infty} \beta^{n-1} c(X_n) \mid X_1 = j\right] P_{ij}$$

Stationary

$$\overset{\downarrow}{=} c(i) + \beta \sum_{j=0}^{\infty} E\left[\sum_{k=0}^{\infty} \beta^k c(X_k) \mid X_0 = j\right] P_{ij}$$

$$= c(i) + \beta \sum_{j=0}^{\infty} h_j P_{ij}$$

Note: In the missing theorem, $g(y_1, y_2, \ldots) = c(j) + \sum_{n=1}^{\infty} \beta^n c(y_n)$. By stationarity, the conditional distribution of $g(X_2, X_3, \ldots)$ given $X_1 = j$ equals that of $g(X_1, X_2, \ldots)$ given $X_0 = j$, the conditional expectations are also equal.

The "Missing Theorem"

Let $X_0, X_1, \ldots$ be a stationary Markov chain. Then for any function g,

$$Pr\{g(X_n, X_{n+1}, \ldots) = x | X_0 = i_0, X_1 = i_1, \ldots X_{n-1} = i\} = Pr\{g(X_n, X_{n+1}, \ldots) = x | X_{n-1} = i\} .$$

Because the Markov chain is stationary, this probability does not depend on n. Also, for your entertainment the function has to be measurable, but of course I did not say that.

Our text actually assumes this in quite a few places without saying so explicitly. For example, time until "eventual" absorbtion is definitely a function of an infinite number of values in the future. Is it so obvious that you can say $Pr\{X_T = 0 | X_0 = i_0, X_1 = i_1, \ldots X_{n-1} = i\} = Pr\{X_T = 0 | X_{n-1} = i\}$ by the Markov property as stated in our book? Not to me.