Corollary If $i \leftrightarrow j$ & i is recurrent, then j is recurrent

Proof Assume i recurrent. Will show $\sum_{k=0}^{\infty} P_{jj}^{(k)} = \infty$
Since $i \leftrightarrow j$, $\exists\, m$ & $n \ni P_{ij}^{(n)} > 0$ & $P_{ji}^{(m)} > 0$

Let $v > 0$. Have

$$\underset{\sim}{P}^{m+v+n} = \underset{\sim}{P}^{m}\,\underset{\sim}{P}^{v}\,\underset{\sim}{P}^{n}, \text{ so}$$

$$\underbrace{P_{jj}^{(m+v+n)} = \sum_{l=0}^{\infty} \underbrace{\left[\sum_{k=0}^{\infty} P_{jk}^{(m)} P_{kl}^{(v)}\right]}_{\text{Element } j,l \text{ of } \underset{\sim}{P}^{(m+v)}} P_{lj}^{(n)}}_{\text{Element } jj \text{ of } P^{m+n+v}}$$

$$\ge \sum_{l=0}^{\infty} \underbrace{\left(P_{ji}^{(m)} P_{il}^{(v)}\right)}_{\text{choose } k=i} P_{lj}^{(n)} = P_{ji}^{(m)} \sum_{l=0}^{\infty} P_{il}^{(v)} P_{lj}^{(n)}$$

$$\ge \underbrace{P_{ji}^{(m)} P_{ii}^{(v)} P_{ij}^{(n)}}_{\text{choose } l=j}$$

Now

$$\sum_{k=0}^{\infty} P_{jj}^{(k)} \ge \sum_{v=0}^{\infty} P_{jj}^{(m+n+v)} \ge \sum_{v=0}^{\infty} P_{ji}^{(m)} P_{ii}^{(v)} P_{ij}^{(n)}$$

$$= P_{ji}^{(m)} P_{ij}^{(n)} \sum_{v=0}^{\infty} P_{ii}^{(v)} = \infty$$

If i is recurrent $\Rightarrow$ j recurrent. If j recurrent then i recurrent