STA 347F2003 Final Formula Sheet

Sums

- If $0 < a < 1$ then $\sum_{k=j}^{\infty} a^k = \frac{a^j}{1-a}$.
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.
- $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

Expected Value: If the discrete random variable Z takes values 0, 1, ..., then $E[Z] = \sum_{k=1}^{\infty} Pr\{Z \geq k\}$.

Vocabulary A state j is said to be **accessible** from i if $P_{ij}^{(n)} > 0$ for some n. Two states that are accessible to one another are said to **communicate**, and we write $i \leftrightarrow j$. All the states that communicate with one another are grouped together in an **equivalence class**. When the state space has only one equivalence class, it is said to be **irreducible**. The **period** of a state i, written $d(i)$, is the greatest common divisor of all the integers $n \geq 1$ such that $P_{ii}^{(n)} > 0$. If $P_{ii} > 0$, then $d(i) = 1$. All states in an equivalence class have the same period.

For any state i, $f_{ii}^{(n)} = Pr\{X_n = i | X_0 = i, X_1 \neq i, X_2 \neq i, \ldots, X_{n-1} \neq i\}$ is the probability that, starting in state i, the first return to i is at the nth transition. We define $f_{ii}^{(0)} = 0$.

$f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)}$ is the probability of *ever* returning to state i. If $f_{ii} = 1$, the state i is said to be **recurrent**. If $f_{ii} < 1$, then i is said to be **transient**. Let i be a recurrent state; the *return time* is $R_i = \min\{n \geq 1 : X_n = i\}$, with $Pr\{R_i = n | X_0 = i\} = f_{ii}^{(n)}$. If $E[R_i | X_0 = i] = \sum_{n=1}^{\infty} n f_{ii}^{(n)} < \infty$, the state i is called **positive recurrent**. Otherwise, i is called **null recurrent**.

Theorem 3.0 $P_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)}$. This is Eq. (3.2) in our text.

Theorem 3.1 State i is transient if and only if $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$.

Corollary 3.1 If $i \leftrightarrow j$, then states i and j are either both transient or both recurrent.

Theorem 4.1 For an irreducible, recurrent, aperiodic, stationary Markov chain, $\lim_{n\to\infty} P_{jj}^{(n)} = \lim_{n\to\infty} P_{ij}^{(n)} = \frac{1}{m_i}$, where $m_i = \sum_{n=1}^{\infty} n f_{ii}^{(n)} = E[R_i | X_0 = i]$.

Theorem 4.2 For an irreducible, positive recurrent, aperiodic, stationary Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ is uniquely determined by $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ and $\sum_{j=0}^{\infty} \pi_j = 1$.

Binomial: $X \sim B(n,p)$ means $Pr\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, , \ldots, n$. Note: $E[X] = np$, $V[X] = np(1-p)$.

Poisson: $X \sim P(\mu)$ means $Pr\{X = k\} = \frac{e^{-\mu}\mu^k}{k!}$ for $k = 0, 1, \ldots$. Note: $E[X] = V[X] = \mu$

Geometric: $X \sim \text{Geometric}\,(a)$ means $Pr\{X = n\} = a(1-a)^{n-1}$ for $n = 1, \ldots$.. Note: $E[X] = 1/a$

Exponential: $X \sim \exp(\lambda)$ means $f_X(x) = \lambda e^{-\lambda x} \mathbf{1}\{x \geq 0\}$. Note $E[X] = 1/\lambda$, and $F_X(x) = (1 - e^{-\lambda x})\mathbf{1}\{x \geq 0\}$.

Gamma: $X \sim G(n, \lambda)$ means $f_X(x) = \frac{\lambda^n}{(n-1)!} e^{-\lambda x} x^{n-1} \mathbf{1}\{x \geq 0\}$. Note $E[X] = \frac{n}{\lambda}$. The sum of n independent exponential random variables is Gamma.