STA 347F2003 Formula Sheet 1

Geometric Series: If $0 < a < 1$ then $\sum_{k=j}^{\infty} a^k = \frac{a^j}{1-a}$.

Expected Value: If the discrete random variable Z takes values 0, 1, …, then $E[Z] = \sum_{k=1}^{\infty} Pr\{Z \geq k\}$.

Vocabulary A state j is said to be **accessible** from i if $P_{ij}^{(n)} > 0$ for some n. Two states that are accessible to one another are said to **communicate**, and we write $i \leftrightarrow j$. All the states that communicate with one another are grouped together in an **equivalence class**. When the state space has only one equivalence class, it is said to be **irreducible**. The **period** of a state j, written $d(j)$, is the greatest common divisor of all the integers $n \geq 1$ such that $P_{ii}^{(n)} > 0$. If $P_{ii} > 0$, then $d(i) = 1$. All states in an equivalence class have the same period.

For any state i, $f_{ii}^{(n)} = Pr\{X_n = i | X_0 = i, X_1 \neq i, X_2 \neq i, \ldots, X_{n-1} \neq i\}$ is the probability that, starting in state i, the first return to i is at the nth transition. We define $f_{ii}^{(0)} = 0$.

$f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)}$ is the probability of *ever* returning to state i. If $f_{ii} = 1$, the state i is said to be **recurrent**. If $f_{ii} < 1$, then i is said to be **transient**. Let i be a recurrent state; the *return time* is $R_i = \min\{n \geq 1 : X_n = i\}$, with $Pr\{R_i = n | X_0 = i\} = f_{ii}^{(n)}$. If $E[R_i | X_0 = i] = \sum_{n=1}^{\infty} n f_{ii}^{(n)} < \infty$, the state i is called **positive recurrent**. Otherwise, i is called **null recurrent**.

Theorem 3.0 $P_{ii}^{(n)} = \sum_{k=0}^{n} f_{ii}^{(k)} P_{ii}^{(n-k)}$. This is Eq. (3.2) in our text.

Theorem 3.1 State i is transient if and only if $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$.

Corollary 3.1 If $i \leftrightarrow j$, then states i and j are either both transient or both recurrent.

Theorem 4.1 For an irreducible, recurrent, aperiodic, stationary Markov chain, $\lim_{n\to\infty} P_{jj}^{(n)} = \lim_{n\to\infty} P_{ij}^{(n)} = \frac{1}{m_i}$, where $m_i = \sum_{n=1}^{\infty} n f_{ii}^{(n)} = E[R_i | X_0 = i]$.

Theorem 4.2 For an irreducible, positive recurrent, aperiodic, stationary Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ is uniquely determined by $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ and $\sum_{j=0}^{\infty} \pi_j = 1$.