STA 347F2003 Assignment 1

Do this assignment in preparation for the quiz on Friday, Sept. 19th. It is not to be handed in.

1. Show that if $Pr(B) > 0$, then $Pr(A \cap B) = Pr(A|B)Pr(B)$.

2. A family has two children. Find the probability that both are girls given that

   (a) The younger child is a girl. [Answer: 1/2]

   (b) At least one is a girl. [Answer: 1/3]

3. Let a pair of fair dice be tossed. If the sum is 6, find the probability that one of the dice is a 2. [Answer: 2/5]

4. Suppose $\{A_1, A_2, \ldots\}$ is *not* a partition of the sample space. Is it still true that $Pr(B) = \sum_{k=1}^{\infty} Pr(B|A_k)Pr(A_k)$? If it is true, prove it. If it is false, give a simple counter-example, preferably one involving a finite number of sets. Begin your answer with the words "The statement is true," or "The statement is false."

   Please note that in this kind of question, you can show a statement to be false (that is, not true in general) by providing a numerical example. However, if a statement is true and all you do is give a numerical example, it is worth *zero* marks.

5. We are given three jars, as follows:

   - Jar A has 5 red balls and 5 blue balls.
   - Jar B has 3 red balls and 2 blue balls.
   - Jar C has 6 red balls and no blue balls.

   We select a jar at random and then draw a ball at random from that jar. If the ball is red, we roll a single fair die. If the ball is blue, we roll two fair dice.

   (a) What is the probability of getting a 4 given that a red ball was chosen? [Answer: 1/6]

   (b) What is the probability of getting a 4 given that a blue ball was chosen? [Answer: 1/12]

   (c) What is the probability of choosing a blue ball? [Answer: 3/10]

   (d) What is the probability of getting a 4? [Answer: 17/120]

   (e) What is the probability of choosing a blue ball given a 4? [Answer: 3/17]

   (f) What is the probability of choosing Jar A given a 4? [Answer: 5/17]

6. Suppose $Pr(A) > 0$. Either show that $Pr(B) = Pr(B|A)Pr(A) + Pr(B^c|A)Pr(A)$ is true in general, or show it is false by giving a counter-example. Begin your answer with the words "The statement is true," or "The statement is false."

7. Let the random variable X take on the values $\{x_1, x_2, \ldots\}$ with non-zero probability, and let the random variable Y take on the values $\{y_1, y_2, \ldots\}$ with non-zero probability. Show that

$$Pr\{X = x_k|Y = y_j\} = \frac{Pr\{Y = y_j|X = x_k\}Pr\{X = x_k\}}{\sum_{n=1}^{\infty} Pr\{Y = y_j|X = x_n\}Pr\{X = x_n\}}$$

8. Suppose $0 < Pr(A) < 1$. Either show that $Pr(B) = Pr(B|A)Pr(A)+Pr(B|A^c)Pr(A^c)$ is true in general, or show it is false by giving a counter-example. Begin your answer with the words "The statement is true," or "The statement is false."

9. We are given three boxes, as follows:

   - Box I has 10 lightbulbs of which 4 are defective.
   - Box II has 6 lightbulbs of which 1 is defective.
   - Box III has 8 lightbulbs of which 3 are defective.

   We select a box at random and then draw a bulb at random. What is the probability that the bulb is defective? [Answer: 113/360]

10. Three machines A, B and C produce respectively 50%, 30% and 20% of the total number of items of a factory. The percentages of defective output of these machines are 3%, 4% and 5%. Suppose a lightbulb is selected and found to be defective. Find the probability that it was produced by machine A. [Answer: 15/37]

11. Let the random variable X take on the values $\{x_1, x_2, \ldots\}$ with non-zero probability, and let the random variable Y take on the values $\{y_1, y_2, \ldots\}$ with non-zero probability. The expected value of X is defined by $E[X] = \sum_{n=1}^{\infty} x_n Pr\{X = x_n\}$. If $T = g(X)$, a formula for the expected value of T is

$$E[T] = E[g(X)] = \sum_{n=1}^{\infty} g(x_n)Pr\{X = x_n\}. \tag{1}$$

The conditional expected value of X given $Y = y$ is defined by

$$E[X|Y = y] = \sum_{n=1}^{\infty} x_n Pr\{X = x_n|Y = y\}. \tag{2}$$

Expression (2) is clearly a function – a function of the argument y. We can make the conditional expectation into a random variable by defining it as this same function of the random variable Y. That is, $E[X|Y = y]$ is just another function $g(y)$, and we are interested in the random variable $g(Y)$. Thus, we define

$$E[X|Y] = \sum_{n=1}^{\infty} x_n Pr\{X = x_n|Y = Y\}. \tag{3}$$

Note that literally, all we are doing here is to replace little y with big Y in the formula for the conditional probability $Pr\{X = x_n|Y = y\}$.

The discussion above establishes that $E[X|Y]$ is just another random variable, and it is reasonable to take its expected value. Guided by expression (1), show that

$E[X] = E[E[X|Y]]$, exchanging order of summation without comment (it's valid by Fubini's theorem, even for infinite sums). This is the famous *double expectation formula.*

12. A motorist will make an insurance claim during a given month with probability p, independently of past events. Once the motorist makes a claim, the policy is cancelled. Let X denote the month in which the claim is made (so that, if the claim comes during the second month, then $X = 2$). Show that the expected month in which the claim is made claim equals $\frac{1}{p}$. Hint: Condition on what happens during month one, for example letting $Y = 1$ if there is a claim during the first month, and $Y = 0$ otherwise. Do you agree that $E[X|Y = 0] = E[X] + 1$?

13. In the preceding example, suppose that the mean amount that the insurance company must pay when a claim is made is a random variable T independent of X (so that $E[T|X] = E[T]$), with $E[T] = M$ dollars. What (average) amount should the insurance company charge per month so that the expected profit for a customer is k dollars? Bear in mind that the policy is cancelled after the customer makes one claim; the game is over once a claim is made.

14. A miner is trapped in a mine containing three doors. Behind the first door is a tunnel which leads to safety after two hours' travel. The second door leads to a tunnel which returns the miner to the choice point after three hours. The third door leads to a tunnel which returns the miner to the choice point after five hours. Assuming the miner is at all times equally likely to choose any one of the three doors, what is the expected length of time until the miner reaches safety? Hint: Condition on what happens the first time. The answer is 10 hours.