STA 312 F23 Assignment 2

(1) See Assignment One.

(2) (a) $\hat{v}_n = \dfrac{-1}{\ell''(\hat{\tau})} = \dfrac{-1}{\frac{-n}{\hat{\tau}^2} - \Sigma x_i^2}$

$= \dfrac{1}{\frac{n}{\hat{\tau}^2} + \Sigma x_i^2}$ with $\hat{\tau} = \sqrt{\dfrac{n}{\Sigma x_i^2}}$

$= \dfrac{1}{n/\left(\frac{n}{\Sigma x_i^2}\right) + \Sigma x_i^2} = \dfrac{1}{\Sigma x_i^2 + \Sigma x_i^2} = \dfrac{1}{2\,\Sigma x_i^2}$

(b) 0.5/sum(x^2) = 0.026

(3) $E(X) = \sum_x x\, p(x) = 0 \cdot (1-\theta) + 1 \cdot \theta = \theta$

$E(X^2) = \sum_x x^2 p(x) = 0^2 (1-\theta) + 1^2 \cdot \theta = \theta$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \theta - \theta^2 = \theta(1-\theta)$

(4) $E(X) = \int_0^\infty x \lambda e^{-\lambda x}\, dx$

$$= \lambda \frac{1!}{\lambda^2} \underbrace{\int_0^\infty \frac{\lambda^2}{\Gamma(2)} e^{-\lambda x} x^{2-1}\, dx}_{=1 \quad \text{GAMMA DENSITY}} = \frac{1}{\lambda}$$

$$E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx$$

$$= \lambda \frac{2!}{\lambda^3} \underbrace{\int_0^\infty \frac{\lambda^3}{\Gamma(3)} e^{-\lambda x} x^{3-1}\, dx}_{=1}$$

$$= 2/\lambda^2$$

$$\mathrm{Var}(X) = E(X^2) - (EX)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

(5) $$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} P(Z \le z)$$

$$= \frac{d}{dz} P\left(\frac{X-\mu}{\sigma} \le z\right) = \frac{d}{dz} P(X \le \sigma z + \mu)$$

$$= \frac{d}{dz} F_X(\sigma z + \mu) = f_X(\sigma z + \mu) \cdot \sigma$$

$$= \cancel{\sigma} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(z\sigma + \cancel{\mu} - \cancel{\mu})^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\cancel{\sigma}^2}\cancel{\sigma}^2 z^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

p. d. f. of $N(0,1)$

6) This could be done with moment-generating functions, but ... Letting $Y = Z^2$, we have for $y \geq 0$

$$f_Y(y) = \frac{d}{dy} P(Y \leq y) = \frac{d}{dy} P(Z^2 \leq y)$$

$$= \frac{d}{dy} P(|Z| \leq y^{\frac{1}{2}}) = \frac{d}{dy} P(-y^{\frac{1}{2}} \leq Z \leq y^{\frac{1}{2}})$$

$$= \frac{d}{dy} \left(F_Z(y^{\frac{1}{2}}) - F_Z(-y^{\frac{1}{2}}) \right)$$

$$= f_Z(y^{\frac{2}{2}}) \cdot \frac{1}{2} y^{-\frac{1}{2}} - f_Z(-y^{\frac{1}{2}})(-1) \frac{1}{2} y^{-\frac{1}{2}}$$

$$= \frac{1}{2} y^{-\frac{1}{2}} \left(f_Z(y^{\frac{1}{2}}) + f_Z(-y^{\frac{1}{2}}) \right)$$

since the standard normal density is symmetric around zero

$$= \cancel{\frac{1}{2}} y^{-\frac{1}{2}} \cancel{2} f_Z(y^{\frac{1}{2}}) = y^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y^{\frac{1}{2}})^2}$$

since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$= \frac{1}{2^{1/2} \Gamma(\frac{1}{2})} e^{-y/2} y^{\frac{1}{2}-1} \quad \text{for } y > 0$$

Density of a Gamma, with $\alpha = \frac{1}{2}$ and Beta $= \frac{1}{2}$

Chi-squared is Gamma $(\alpha = \frac{\nu}{2}, \beta = \frac{1}{2})$, so this is $\chi^2(1)$.

7 (a) $E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$

$= \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{1}{n} n\mu = \mu$

(b) $Var(\bar{X}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$

$= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$

$\overset{ind}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2}\sum_{i=1}\sigma^2$

$= \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$

(8) (a) $$\ell'(\lambda) = \frac{d}{d\lambda} \log \prod_{i=1}^{n} \left(\frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \right)$$

$$= \frac{d}{d\lambda} \log \left(\frac{e^{-n\lambda} \lambda^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!} \right)$$

$$= \frac{d}{d\lambda} \left(-n\lambda + \left(\sum_{i=1}^{n} x_i\right) \log \lambda - \log \prod_{i=1}^{n} x_i! \right)$$

$$= -n + \frac{\Sigma x_i}{\lambda} + 0 = -n + (\Sigma x_i)\lambda^{-1}$$

$$\stackrel{\text{set}}{=} 0 \Rightarrow n = \frac{\sum_{i=1}^{n} x_i}{\lambda} \Rightarrow \lambda = \frac{\Sigma x_i}{n} = \bar{x}$$

(b) $$\ell''(\lambda) = \frac{d}{d\lambda}\left(-n + (\Sigma x_i)\lambda^{-1}\right) = (-1)(\Sigma x_i)\lambda^{-2}$$

$$= \frac{-\Sigma x_i}{\lambda^2} < 0 \quad \text{CCD} \quad \frown \quad \text{MAX}$$

(c) $\bar{x} = 4.2$

(8d) Using Problem 7, $Var(\hat{\lambda}) = var(\bar{x}) = \frac{\sigma^2}{n} = \frac{\lambda}{n}$

So $\bar{X}_n \dot{\sim} N(\lambda, \frac{\lambda}{n})$, and

$$Z = \frac{\bar{x} - \lambda}{\sqrt{\hat{\lambda}/n}} \dot{\sim} N(0,1)$$, so CI is

$$\bar{x} \pm 1.96 \sqrt{\bar{x}/n} = 4.2 \pm 1.96 \sqrt{4.2/49}$$

$$= 4.2 \pm 0.574 = (3.63, 4.77)$$

(e) (i) $z_{0.025} = 1.96$

(ii) $$Z = \frac{\bar{x} - \lambda_0}{\sqrt{\bar{x}/n}} = \frac{\sqrt{n}(\bar{x} - \lambda_0)}{\sqrt{\bar{x}}} = \frac{\sqrt{49}(4.2-3)}{\sqrt{4.2}}$$

$$= 4.099 \approx 4.1$$, or

$$Z = \frac{\sqrt{n}(\bar{x} - \lambda_0)}{\sqrt{\lambda_0}} = \frac{\sqrt{49}(4.2-3)}{\sqrt{3}} = 4.85$$

(iii) Yes, (iv) yes

(v) Conclude $\lambda > 3$

(vi) 2*(1-pnorm(4.1)) = 4.1315 e-05

9 (a) $\int_0^\infty \pi e^{-\pi/x} \frac{1}{x^2}\, dx$

Let $u = \frac{1}{x} = x^{-1}$ $du = (-1)x^{-2}$ $= \frac{-1}{x^2}\, dx$

x	u
∞	0
0	∞

$$= -\int_\infty^0 \pi e^{-\pi u}\, du = \int_0^\infty \pi e^{-\pi u}\, du = 1$$

Because it's an exponential density

(b) $\ell'(\pi) = \frac{d}{d\pi} \log \prod_{i=1}^n \pi e^{-\pi/x_i} \frac{1}{x_i^2}$

$$= \frac{d}{d\pi} \log \left(\pi^n e^{-\pi \sum_{i=1}^n \frac{1}{x_i}} \frac{1}{\prod_{i=1}^n x_i^2} \right)$$

$$= \frac{d}{d\pi} \left(n \log \pi - \pi \sum_{i=1}^n \frac{1}{x_i} - \sum_{i=1}^n 2 \log x_i \right)$$

$$= \frac{n}{\pi} - \sum_{i=1}^n \frac{1}{x_i} = n\pi^{-1} - \sum_{i=1}^n \frac{1}{x_i}$$

$$\overset{set}{=} 0 \Rightarrow \frac{n}{\pi} = \sum_{i=1}^n \frac{1}{x_i} \Longrightarrow \hat{\pi} = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}$$

2nd der test

$$\ell''(\pi) = n(-1)\pi^{-2} - 0 = \frac{-n}{\pi^2} < 0 \quad \text{CCD} \ \cap$$

MAX

(c) $\hat{v}_n = \frac{-1}{\ell''(\hat{\pi})} = \frac{\hat{\pi}^2}{n}$

(9d) This is R work.

i. $\hat{\pi}_n = 2.74054$1

ii. 95% CI: (2.43, 3.05

(e) More R work

i. ± 1.96

ii. $Z = -2.53$

iii. 2*(1-pnorm(abs(Z))) = 0.01125527

iv. Yes reject Ho

v. Yes significant

vi. Yes contradicts Ho: $\pi = \pi$ ⌣̈

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> x = scan("http://www.utstat.toronto.edu/brunner/data/legal/inversegamma.data.txt")
Read 300 items
> n = length(x)
> pihat = 1/mean(1/x); pihat
[1] 2.740541
> se_pihat = sqrt(pihat^2/n); se_pihat
[1] 0.1582252
> CI = c(pihat-1.96*se_pihat, pihat+1.96*se_pihat); CI
[1] 2.430420 3.050662
> Z = (pihat-3.14159)/se_pihat; Z
[1] -2.534672
> pvalue = 2 * (1-pnorm(abs(Z))); pvalue
[1] 0.01125527
>
>
```