Poisson Regression

Poisson Distribution

$$P_r(Y=y) = \frac{e^{-\lambda}\lambda^{y}}{y!}, \text{ for } y = 0, 1, 2 \ldots$$

$$\lambda > 0$$

$$E(Y) = \lambda = Var(Y)$$

Good model for counts

Linear model for $\log \lambda = \ln \lambda$

$$\log \lambda = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k = x'\beta$$

$$\iff \lambda = e^{x'\beta}$$

(Implicitly for $i = 1, \ldots, n$

$$l(\beta) = \prod_{i=1}^{n} \frac{e^{-e^{x_i'\beta}} \left(e^{x_i'\beta}\right)^{y_i}}{y_i!}$$

$$= \frac{e^{-\sum_{i=1}^{n} x_i'\beta} \, e^{\sum_{i=1}^{n} x_i'\beta y_i}}{\prod_{i=1}^{n} y_i!}$$

$$\log l(\beta) = -\sum_{i=1}^{n} x_i'\beta + \sum_{i=1}^{n} e^{x_i'\beta y_i} \underbrace{- \sum_{i=1}^{n} \log(y_i!)}_{c}$$

$$\lambda = e^{x'\beta}$$

$$\ell(\beta) = \prod_{i=1}^{n} \frac{e^{-e^{x_i'\beta}} e^{x_i'\beta y_i}}{y_i!}$$

$$= \frac{e^{-\sum_{i=1}^{n} e^{x_i'\beta}} e^{\sum_{i=1}^{n} x_i'\beta y_i}}{\prod_{i=1}^{n} \log y_i!}$$

$\log \ell(\beta)$

$$= -\sum_{i=1}^{n} e^{x_i'\beta} + \sum_{i=1}^{n} x_i'\beta y_i - c$$

Differentiate with respect to $\beta_0, \beta_1, -- \beta_k$
set $= 0$, can't solve

Numerical Maximum Likelihood

Get $\hat{\beta}_n \; \dot{\sim} \; N_{k+1}(\beta, V_n)$

Different asymptotic var-var matrix
V cov(model)

Likelihood Ratio tests

Wald tests
etc.

Recall the jobs study

- 106 employed in job related to field
- 74 " " unrelated " "
- 20 unemployed.

Independent Poisson proceses,

Rates λ_1, λ_2 λ_3

Conditionally on n (# of students)

Multinomial

$$\pi_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$\pi_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3}$$

$$\pi_3 = \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}$$

But Let's do poisson regression

$\log \lambda = \beta_0 + \beta_1 d_1 + \beta_2 d_2$

	d_1	d_2	$\lambda = E(Y)$
Related	0	0	e^{β_0}
Unrelated	1	0	$e^{\beta_0} e^{\beta_1}$
Unemp	0	1	$e^{\beta_0} e^{\beta_2}$

On average, expect $\underline{e^{\beta_2}}$ times as many unemployed as students with jobs related to field of study.

$$\frac{e^{\beta_0} e^{\beta_2}}{e^{\beta_0}}$$

In general, increase ~~variable~~ x_j by one unit, holding all other vars constant
Mutiply Expected Y by e^{β_j}.

$$\frac{e^{\beta_0} e^{\beta_1 (x_1 + 1)} e^{\beta_2 x_2}}{e^{\beta_0} e^{\beta_1 x_1} e^{\beta_2 x_2}} = e^{\beta_1}$$