

# Logistic Regression Part One

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# Suggested Reading

- In Ch. 3, skim pages 65-68; we'll come back to generalized linear models later.
- Read pages 68-73 about logistic regression.
- Read Chapter 4.
- The discussion of dummy variables in lecture may be clearer than the text's.
- Exactly what do you need to know? See homework problems.

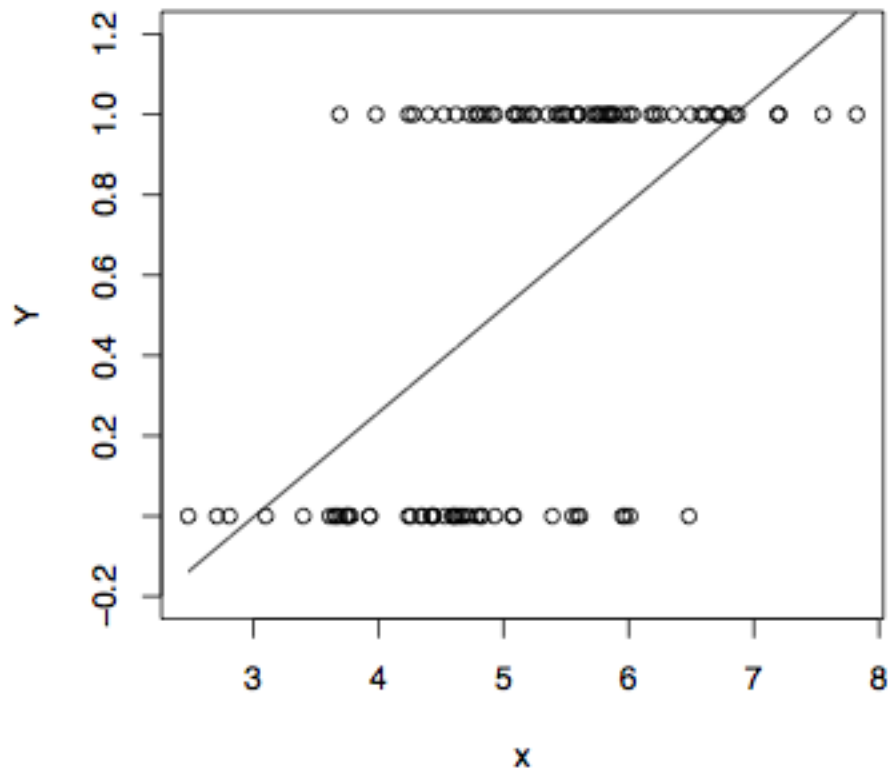
# Logistic Regression

For a binary dependent variable:  
1=Yes, 0=No

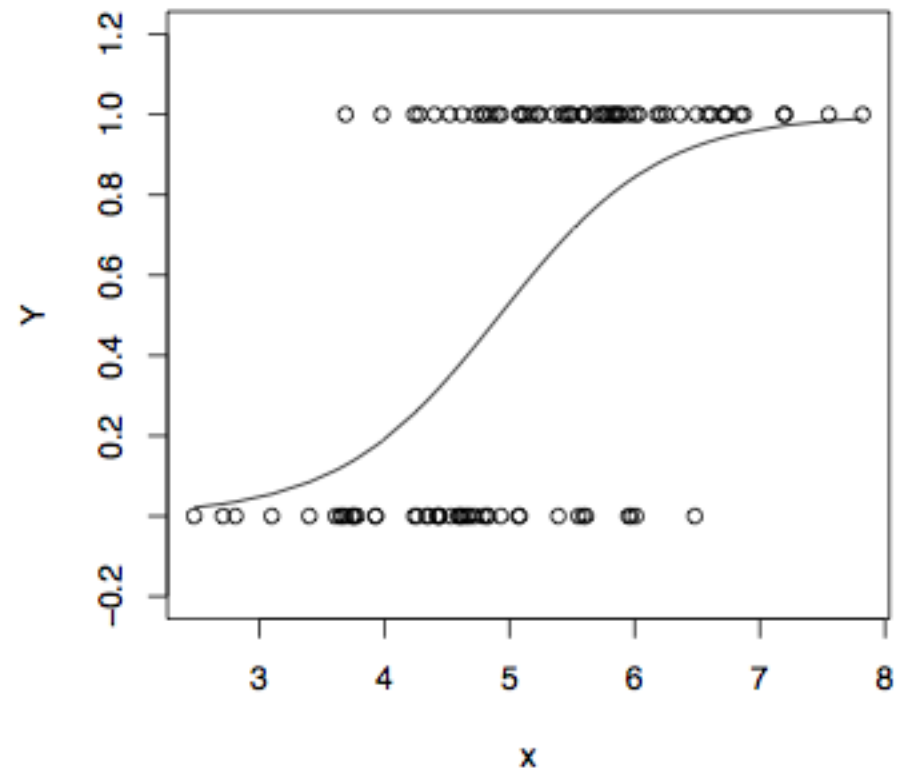
$$Pr\{Y = 1 | \mathbf{X} = \mathbf{x}\} = \pi$$

# Least Squares vs. Logistic Regression

Least Squares Line



Logistic Regression Curve



# Linear regression model for the log odds of the event $Y=1$

$$\log \left( \frac{\pi}{1 - \pi} \right) = \beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1}$$

Note  $\pi$  is a *conditional* probability.

# Equivalent Statements

$$\log \left( \frac{\pi}{1 - \pi} \right) = \beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1}$$

$$\begin{aligned} \frac{\pi}{1 - \pi} &= e^{\beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1}} \\ &= e^{\beta_0} e^{\beta_1 x_1} \dots e^{\beta_{p-1} x_{p-1}}, \end{aligned}$$

$$\pi = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1}}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1}}}.$$

$$E(Y|\mathbf{x}) = \pi = \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1}}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1}}}$$

- A distinctly non-linear function
- Non-linear in the betas
- So logistic regression is an example of *non-linear regression*.

$F(x) = \frac{e^x}{1+e^x}$  is called the *logistic distribution*.

- Could use any cumulative distribution function:

$$\pi = F(\beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1})$$

- CDF of the standard normal used to be popular
- Called probit analysis
- Can be closely approximated with a logistic regression.



In terms of log odds, logistic regression is like regular regression

$$\log \left( \frac{\pi}{1 - \pi} \right) = \beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1}$$

## In terms of plain odds,

- (Exponential function of) the logistic regression coefficients are *odds ratios*
- For example, “Among 50 year old men, the odds of being dead before age 60 are three times as great for smokers.”

$$\frac{\text{Odds of death given smoker}}{\text{Odds of death given nonsmoker}} = 3$$

# Logistic regression

- $X=1$  means smoker,  $X=0$  means non-smoker
- $Y=1$  means dead,  $Y=0$  means alive
- Log odds of death =  $\beta_0 + \beta_1 x$
- Odds of death =  $e^{\beta_0} e^{\beta_1 x}$

$$\text{Odds of Death} = e^{\beta_0} e^{\beta_1 x}$$

<b>Group</b>	$x$	<b>Odds of Death</b>
Smokers	1	$e^{\beta_0} e^{\beta_1}$
Non-smokers	0	$e^{\beta_0}$

$$\frac{\text{Odds of death given smoker}}{\text{Odds of death given nonsmoker}} = \frac{e^{\beta_0} e^{\beta_1}}{e^{\beta_0}} = e^{\beta_1}$$

# Cancer Therapy Example

$$\text{Log Survival Odds} = \beta_0 + \beta_1 d_1 + \beta_2 d_2 + \beta_3 x$$

Treatment	$d_1$	$d_2$	Odds of Survival = $e^{\beta_0} e^{\beta_1 d_1} e^{\beta_2 d_2} e^{\beta_3 x}$
Chemotherapy	1	0	$e^{\beta_0} e^{\beta_1} e^{\beta_3 x}$
Radiation	0	1	$e^{\beta_0} e^{\beta_2} e^{\beta_3 x}$
Both	0	0	$e^{\beta_0} e^{\beta_3 x}$

$x$  is severity of disease

For any given disease severity  $x$ ,

$$\frac{\text{Survival odds with Chemo}}{\text{Survival odds with Both}} = \frac{e^{\beta_0} e^{\beta_1} e^{\beta_3 x}}{e^{\beta_0} e^{\beta_3 x}} = e^{\beta_1}$$

# In general,

- When  $x_k$  is increased by one unit and all other independent variables are held constant, the odds of  $Y=1$  are multiplied by  $e^{\beta_k}$
- That is,  $e^{\beta_k}$  is an **odds ratio** --- the ratio of the odds of  $Y=1$  when  $x_k$  is increased by one unit, to the odds of  $Y=1$  when everything is left alone.
- As in ordinary regression, we speak of “controlling” for the other variables.

# The conditional probability of $Y=1$

$$\begin{aligned}\pi &= \frac{e^{\beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1}}}{1 + e^{\beta_0 + \beta_1 x_1 + \dots + \beta_{p-1} x_{p-1}}} \\ &= \frac{e^{\mathbf{x}'_i \boldsymbol{\beta}}}{1 + e^{\mathbf{x}'_i \boldsymbol{\beta}}}\end{aligned}$$

This formula can be used to calculate a predicted  $P(Y=1|\mathbf{x})$ . Just replace betas by their estimates

It can also be used to calculate the probability of getting the sample data values we actually did observe, as a function of the betas.



# Likelihood Function

$$\begin{aligned}\ell(\boldsymbol{\beta}) &= \prod_{i=1}^n P(Y_i = y_i | \mathbf{x}_i) = \prod_{i=1}^n \pi^{y_i} (1 - \pi)^{1-y_i} \\ &= \prod_{i=1}^n \left( \frac{e^{\mathbf{x}'_i \boldsymbol{\beta}}}{1 + e^{\mathbf{x}'_i \boldsymbol{\beta}}} \right)^{y_i} \left( 1 - \frac{e^{\mathbf{x}'_i \boldsymbol{\beta}}}{1 + e^{\mathbf{x}'_i \boldsymbol{\beta}}} \right)^{1-y_i} \\ &= \prod_{i=1}^n \left( \frac{e^{\mathbf{x}'_i \boldsymbol{\beta}}}{1 + e^{\mathbf{x}'_i \boldsymbol{\beta}}} \right)^{y_i} \left( \frac{1}{1 + e^{\mathbf{x}'_i \boldsymbol{\beta}}} \right)^{1-y_i} \\ &= \prod_{i=1}^n \frac{e^{y_i \mathbf{x}'_i \boldsymbol{\beta}}}{1 + e^{\mathbf{x}'_i \boldsymbol{\beta}}} \\ &= \frac{e^{\sum_{i=1}^n y_i \mathbf{x}'_i \boldsymbol{\beta}}}{\prod_{i=1}^n (1 + e^{\mathbf{x}'_i \boldsymbol{\beta}})}\end{aligned}$$

# Maximum likelihood estimation

- Likelihood = Conditional probability of getting the data values we did observe,
- As a function of the betas
- Maximize the (log) likelihood with respect to betas.
- Maximize numerically (“Iteratively re-weighted least squares”)
- Likelihood ratio tests as usual
- Divide regression coefficients by estimated standard errors to get Z-tests of  $H_0: \beta_j=0$ .
- These Z-tests are like the t-tests in ordinary regression.

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