Wald tests

- MLEs have an approximate multivariate normal sampling distribution for large samples (Thanks Mr. Wald.)
- Approximate mean vector = vector of true parameter values for large samples
- Asymptotic variance-covariance matrix is easy to estimate
- H_0: $\mathbf{C}\boldsymbol{\theta} = \boldsymbol{h}$ (Linear hypothesis)
- For logistic regression, $\boldsymbol{\theta} = \boldsymbol{\beta}$

$$H_0 : \mathbf{C}\boldsymbol{\theta} = \mathbf{h}$$

$$\mathbf{C}\widehat{\boldsymbol{\theta}} - \mathbf{h} \text{ is multivariate normal as } n \rightarrow \infty$$

Leads to a straightforward chisquare test

- Called a Wald test
- Based on the full (maybe even saturated) model
- Asymptotically equivalent to the LR test
- Not as good as LR for smaller samples
- Very convenient, especially with SAS

Example of *H_0: **Cθ=h***

Suppose $\boldsymbol{\theta} = (\theta_1, \ldots \theta_7)$, with

$$H_0 : \theta_1 = \theta_2,\, \theta_6 = \theta_7,\, \frac{1}{3}\left(\theta_1 + \theta_2 + \theta_3\right) = \frac{1}{3}\left(\theta_4 + \theta_5 + \theta_6\right)$$

$$\left[\begin{array}{rrrrrrr} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 0 \end{array}\right] \left[\begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \\ \theta_7 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

Multivariate Normal Facts

$$\mathbf{X} \sim N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Rightarrow \mathbf{CX} \sim N_r(\mathbf{C}\boldsymbol{\mu}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}')$$

where $\mathbf{C}$ is $r \times k$, rank r, $r \leq k$.

$$(\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \sim \chi^2(k)$$

$$(\mathbf{CX} - \mathbf{C}\boldsymbol{\mu})'(\mathbf{C}\boldsymbol{\Sigma}\mathbf{C}')^{-1}(\mathbf{CX} - \mathbf{C}\boldsymbol{\mu}) \sim \chi^2(r)$$

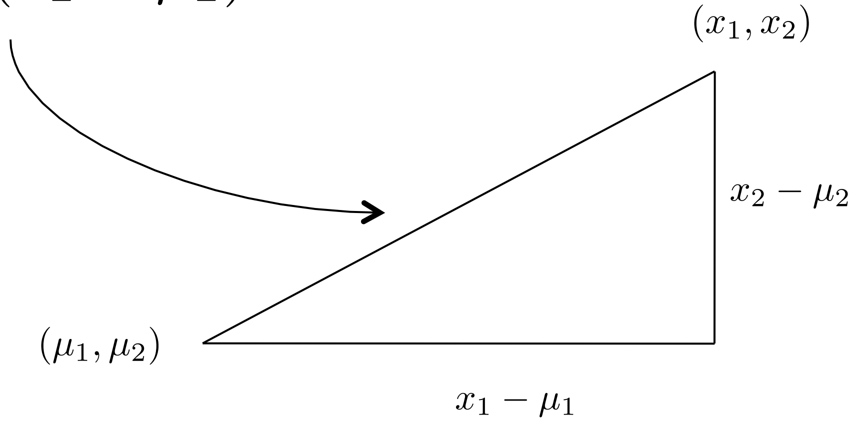
Analogies

- Univariate Normal
  - $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right]$
  - $\frac{(x-\mu)^2}{\sigma^2}$ is the squared Euclidian distance between x and μ, in a space that is stretched by σ^2.
  - $\frac{(X-\mu)^2}{\sigma^2} \sim \chi^2(1)$
- Multivariate Normal
  - $f(\mathbf{x}) = \frac{1}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}(2\pi)^{\frac{k}{2}}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$
  - $(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ is the squared Euclidian distance between $\mathbf{x}$ and $\boldsymbol{\mu}$, in a space that is warped and stretched by $\boldsymbol{\Sigma}$.
  - $(\mathbf{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu}) \sim \chi^2(k)$

Distance: Suppose **Σ = I**$_2$

$$\begin{aligned}
d^2 &= (\mathbf{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu}) \\
&= \left[\begin{array}{cc} x_1-\mu_1, & x_2-\mu_2 \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \left[\begin{array}{c} x_1-\mu_1 \\ x_2-\mu_2 \end{array}\right] \\
&= \left[\begin{array}{cc} x_1-\mu_1, & x_2-\mu_2 \end{array}\right] \left[\begin{array}{c} x_1-\mu_1 \\ x_2-\mu_2 \end{array}\right] \\
&= (x_1-\mu_1)^2+(x_2-\mu_2)^2
\end{aligned}$$

$$d = \sqrt{(x_1-\mu_1)^2+(x_2-\mu_2)^2}$$



Approximately, for large *N*

$\widehat{\boldsymbol{\theta}} \sim N_k(\boldsymbol{\theta}, \mathbf{V}(\boldsymbol{\theta})) \qquad \mathbf{C}\widehat{\boldsymbol{\theta}} \sim N_k(\mathbf{C}\boldsymbol{\theta}, \mathbf{CVC}^\prime)$

If $H_0 : \mathbf{C}\boldsymbol{\theta} = \mathbf{h}$ is true,

$(\mathbf{C}\widehat{\boldsymbol{\theta}} - \mathbf{h})^\prime (\mathbf{CVC}^\prime)^{-1} (\mathbf{C}\widehat{\boldsymbol{\theta}} - \mathbf{h}) \sim \chi^2(r)$

$\mathbf{V} = \mathbf{V}(\boldsymbol{\theta})$ is unknown, but

$$\begin{aligned} W &= (\mathbf{C}\widehat{\boldsymbol{\theta}} - \mathbf{h})^\prime (\mathbf{C}\widehat{\mathbf{V}}\mathbf{C}^\prime)^{-1} (\mathbf{C}\widehat{\boldsymbol{\theta}} - \mathbf{h}) \\ &\sim \chi^2(r) \end{aligned}$$

Wald Test Statistic

$$W = (\mathbf{C}\widehat{\boldsymbol{\theta}} - \mathbf{h})'(\mathbf{C}\widehat{\mathbf{V}}\mathbf{C}')^{-1}(\mathbf{C}\widehat{\boldsymbol{\theta}} - \mathbf{h})$$

- Approximately chi-square with $df = r$ for large N if H_0: $\boldsymbol{C\theta}=\boldsymbol{h}$ is true
- Matrix $\boldsymbol{C}$ is r × k, $r \leq k$, rank r
- Matrix $\boldsymbol{V(\theta)}$ is called the “Asymptotic Covariance Matrix” of $\widehat{\boldsymbol{\theta}}$
- $\widehat{\mathbf{V}}$ is the *estimated* Asymptotic Covariance Matrix
- How to calculate $\widehat{\mathbf{V}}$?

Fisher Information Matrix $\mathcal{J}$

- Element (*i,j*) is $-\dfrac{\partial^2}{\partial\theta_i\partial\theta_j}\ell(\boldsymbol{\theta},\mathbf{Y}),\ \text{where}$

- The log likelihood is

$$\ell(\boldsymbol{\theta},\mathbf{Y}) = \sum_{i=1}^{N} \log f(Y_i;\boldsymbol{\theta}).$$

- This is sometimes called the *observed* Fisher information – based on the observed data Y_1, …, Y_N

For a random sample *Y₁*, …, *Y_N* (No *x* values)

- Independent and identically distributed
- Fisher information in a single observation is

$$\boldsymbol{\mathcal{I}}(\boldsymbol{\theta}) = \left[E[-\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log f(Y;\boldsymbol{\theta})] \right]$$

- Estimate expected value with sample mean

$$\widehat{\boldsymbol{\mathcal{I}}}(\boldsymbol{\theta}) = \frac{1}{N}\sum_{i=1}^{N} -\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log f(Y_i;\boldsymbol{\theta})$$

Fisher Information in the whole sample

- $N \cdot \boldsymbol{\mathcal{I}}(\boldsymbol{\theta})$
- Estimate it with the observed information
  $$N \cdot \widehat{\boldsymbol{\mathcal{I}}}(\boldsymbol{\theta}) = \boldsymbol{\mathcal{J}}(\boldsymbol{\theta})$$
- Evaluate this at the MLE and we have a <u>statistic</u>:
  $$\boldsymbol{\mathcal{J}}(\widehat{\boldsymbol{\theta}})$$
- Call it the **Fisher Information.** Technically it’s the observed Fisher information evaluated at the MLE.

For a simple logistic regression

- $\boldsymbol{\theta} = \boldsymbol{\beta} = (\beta_0, \beta_1)$
- $\ell(\boldsymbol{\beta}, \mathbf{y}) = \beta_0 \sum_{i=1}^{N} y_i + \beta_1 \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} \log(1 + e^{\beta_0 + \beta_1 x_i})$

$$\begin{aligned}
\boldsymbol{\mathcal{J}}(\widehat{\boldsymbol{\beta}}) &= -\left[\begin{array}{cc} \frac{\partial^2 \ell}{\partial \beta_0^2} & \frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_1} \\ \frac{\partial^2 \ell}{\partial \beta_1 \partial \beta_0} & \frac{\partial^2 \ell}{\partial \beta_1^2} \end{array}\right]\Bigg|_{\beta_0 = \widehat{\beta}_0, \beta_1 = \widehat{\beta}_1} \\
&= \left[\begin{array}{cc} \sum_{i=1}^{N} \frac{e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i}}{(1+e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i})^2} & \sum_{i=1}^{N} \frac{x_i e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i}}{(1+e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i})^2} \\ \sum_{i=1}^{N} \frac{x_i e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i}}{(1+e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i})^2} & \sum_{i=1}^{N} \frac{x_i^2 e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i}}{(1+e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i})^2} \end{array}\right]
\end{aligned}$$

The asymptotic covariance matrix is the inverse of the Fisher Information

Meaning that the estimated asymptotic covariance matrix of the MLE is the inverse of the observed Fisher information matrix, evaluated at the MLE.

$$\widehat{\mathbf{V}} = \boldsymbol{\mathcal{J}}(\widehat{\boldsymbol{\theta}})^{-1}, \text{ where } \boldsymbol{\mathcal{J}}(\widehat{\boldsymbol{\theta}}) = \left[-\frac{\partial^2}{\partial\theta_i \partial\theta_j} \ell(\boldsymbol{\theta}, \mathbf{Y}) \right] \Big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$

Low Birth Weight Example

$$\mathcal{J}(\widehat{\boldsymbol{\beta}}) = \left[\begin{array}{cc} \sum_{i=1}^{N} \frac{e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i}}{(1+e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i})^2} & \sum_{i=1}^{N} \frac{x_i e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i}}{(1+e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i})^2} \\ \sum_{i=1}^{N} \frac{x_i e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i}}{(1+e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i})^2} & \sum_{i=1}^{N} \frac{x_i^2 e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i}}{(1+e^{\widehat{\beta}_0 + \widehat{\beta}_1 x_i})^2} \end{array} \right]$$

```
> simp = glm(low ~ lwt, family=binomial); simp$coefficients
(Intercept)         lwt
 0.99831432 -0.01405826
> x =  lwt; xb = simp$coefficients[1]+x*simp$coefficients[2]
> kore = exp(xb)/(1+exp(xb))^2
> J = matrix(nrow=2,ncol=2)
> J[1,1] = sum(kore); J[1,2] = sum(x*kore)
> J[2,1]=J[1,2];      J[2,2] = sum(x^2*kore)
> J
           [,1]       [,2]
[1,]   39.38591   4908.917
[2,] 4908.91670 638101.268
```

Compare Outputs

- R

```
> solve(J) # Inverse
             [,1]          [,2]
[1,]  0.616681831 -4.744137e-03
[2,] -0.004744137  3.806382e-05
```

- SAS **proc logistic** output from **covb** option

```
        Estimated Covariance Matrix


Parameter      Intercept           lwt

Intercept       0.616679      -0.00474
lwt             -0.00474      0.000038
```

Connection to Numerical Optimization

- Suppose we are minimizing the minus log likelihood by a direct search.
- We have reached a point where the gradient is close to zero. Is this point a minimum?
- Hessian is a matrix of mixed partial derivatives. If its determinant is positive at a point, the function is concave up there.
- It's *the* multivariable second derivative test.
- The Hessian at the MLE is <u>exactly</u> the Fisher Information:

$$\mathcal{J}(\widehat{\boldsymbol{\theta}}) = \left[\frac{\partial^2}{\partial\theta_i\partial\theta_j} - \ell(\boldsymbol{\theta}, \mathbf{Y})\right]\Bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$

Asymptotic Covariance Matrix $\widehat{\mathbf{V}}$ is Useful

- Square roots of diagonal elements are standard errors – Denominators of Z-test statistics. Also used for confidence intervals.
- Diagonal elements converge to the respective Cramér-Rao lower bounds for the variance of an estimator: “Asymptotic efficiency”
- And of course there are Wald tests

$$W = (\mathbf{C}\widehat{\boldsymbol{\theta}} - \mathbf{h})^\prime (\mathbf{C}\widehat{\mathbf{V}}\mathbf{C}^\prime)^{-1} (\mathbf{C}\widehat{\boldsymbol{\theta}} - \mathbf{h})$$

Score Tests

- $\widehat{\boldsymbol{\theta}}$ is the MLE of $\boldsymbol{\theta}$, size $k \times 1$
- $\widehat{\boldsymbol{\theta}}_0$ is the MLE under H_0, size $k \times 1$
- $\mathbf{u}(\boldsymbol{\theta}) = (\frac{\partial \ell}{\partial \theta_1}, \ldots \frac{\partial \ell}{\partial \theta_k})'$ is the gradient.
- $\mathbf{u}(\widehat{\boldsymbol{\theta}}) = 0$
- If H_0 is true, $\mathbf{u}(\widehat{\boldsymbol{\theta}}_0)$ should also be close to zero.
- Under H_0 for large N, $\mathbf{u}(\widehat{\boldsymbol{\theta}}_0) \sim N_k(\mathbf{0}, \boldsymbol{\mathcal{J}}(\boldsymbol{\theta}))$, approximately.
- And,

$$S = \mathbf{u}(\widehat{\boldsymbol{\theta}}_0)' \boldsymbol{\mathcal{J}}(\widehat{\boldsymbol{\theta}}_0)^{-1} \mathbf{u}(\widehat{\boldsymbol{\theta}}_0) \sim \chi^2(r)$$

Where *r* is the number of restrictions imposed by H_0