The Multivariate Normal Distribution

The $p \times 1$ random vector $\mathbf{X}$ is said to have a *multivariate normal distribution*, and we write $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, if $\mathbf{X}$ has (joint) density

$$f(\mathbf{x}) = \frac{1}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}(2\pi)^{\frac{p}{2}}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right],$$

where $\boldsymbol{\mu}$ is $p \times 1$ and $\boldsymbol{\Sigma}$ is $p \times p$ symmetric and positive definite. We won't explicitly use the positive definite property in this class.

The multivariate normal reduces to the univariate normal when $p = 1$. Properties of the multivariate mormal include the following.

1. $E[\mathbf{X}] = \boldsymbol{\mu}$
2. $\sigma^2\{\mathbf{X}\} = \boldsymbol{\Sigma}$
3. If $\mathbf{c}$ is a vector of constants, $\mathbf{X} + \mathbf{c} \sim N(\mathbf{c} + \boldsymbol{\mu}, \boldsymbol{\Sigma})$
4. If $\mathbf{A}$ is a matrix of constants, $\mathbf{AX} \sim N(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$
5. All the marginals (dimension less than p) of $\mathbf{X}$ are (multivariate) normal, but it is possible to have a collection of univariate normals whose joint distribution is not multivariate normal.
6. For the multivariate normal, zero covariance implies independence. The multivariate normal is the only distribution with this property.
7. The random variable $(\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu})$ has a chi-square distribution with p degrees of freedom.