Interpretation of regression coefficients[1]

STA 302 Fall 2020

[1] See last slide for copyright information.

Average response

The model says

$$E(y) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$$

- Can be viewed as a conditional expected value, given the values $x_1, \ldots, x_k$.
- Theoretically, there is a sub-population for each set of $x_1, \ldots, x_k$ values.
- $E(y|x_1, \ldots, x_k)$ is the sub-population mean (average response) for that sub-population.

$E(y|\mathbf{x}) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$

$$g(x_1, \ldots, x_k) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$$

Examine $g(x_1, \ldots, x_k)$ as a mathematical function, to see what the regression coefficients mean.

Simple regression

$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$

$$g(x) = \beta_0 + \beta_1 x$$

- The equation of a straight line.
- Say x is income and y is credit card debt.
- $\beta_1 > 0$ would mean that higher income tends to go with higher debt, on average.
- Call it a "positive (linear) relationship."
- $\beta_1 < 0$ would mean that higher income tends to go with lower debt, on average.
- Call it a "negative (linear) relationship."
- If the model is correct, $\beta_1 = 0$ would mean that there is no connection at all between income and average credit card debt.
- This is why testing $H_0 : \beta_1 = 0$ is so important.

$$t = \frac{\mathbf{a}'\widehat{\boldsymbol{\beta}} - t_0}{\sqrt{MSE\,\mathbf{a}'(X'X)^{-1}\mathbf{a}}} \stackrel{H_0}{\sim} t(n-k-1)$$

Estimated regression coefficients

$\widehat{E(y|x)} = \widehat{\beta}_0 + \widehat{\beta}_1 x$

- The same talk applies, with the addition of "estimated" or "predicted."
- *Estimated* average credit card debt is higher for consumers with higher incomes (if $\widehat{\beta}_1 > 0$).
- *Predicted* credit card debt is higher for consumers with higher incomes (if $\widehat{\beta}_1 > 0$).
- *Estimated* average credit card debt is lower for consumers with higher incomes (if $\widehat{\beta}_1 < 0$).
- *Predicted* credit card debt is lower for consumers with higher incomes (if $\widehat{\beta}_1 < 0$).
- Suppose annual income is in thousands of dollars. The question says: "When annual income is $1,000 higher, estimated average credit card debt is ________ higher. The answer is a number from your printout." Write the value of $\widehat{\beta}_1$.

Sometimes loose language is okay

- Technically, regression is about the connection between x and *expected*, or *average* y.
- But sometimes people (and my questions) speak just of the relationship between x and y.
- Like the relationship between High School GPA and University GPA.
- Yes, technically $g(x) = \beta_0 + \beta_1 x$ gives the relationship between High School GPA and *average* University GPA.
- But it's harmless – actually it's helpful. If necessary you can clarify.

Plain language is important

- If you can only be understood by mathematicians and statisticians, your knowledge is much less valuable.
- Often a question will say "Give the answer in plain, non-statistical language."
- This means if x is income and y is credit card debt, you make a statement about income and average or predicted credit card debt, like the ones on the preceding slides.
- If you use mathematical notation or words like null hypothesis, unbiased estimator, p-value or statistically significant, you will lose a lot of marks even if the statement is correct. Even avoid "positive relationship," and so on.
- If the study is about fish, talk about fish.
- If the study is about blood pressure, talk about blood pressure.
- If the study is about breaking strength of yarn, talk about breaking strength of yarn.
- Assume you are talking to your boss, who was a Commerce major and does not like to feel stupid.

We will be guided by hypothesis tests with $\alpha = 0.05$

For plain-language conclusions

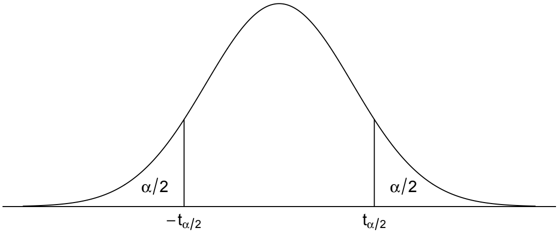
- If we do not reject a null hypothesis like $H_0 : \beta_1 = 0$, we will not draw a definite conclusion.
- Instead, say things like:
  - There is no evidence of a connection between blood sugar level and mood.
  - These results are not strong enough for us to conclude that attractiveness is related to mark in first-year Computer Science.
  - These results are consistent with no effect of dosage level on bone density.
- If the null hypothesis is not rejected, please do *not* claim that the drug has no effect, etc..
- In this we are taking Fisher's side in a historical fight between Fisher on one side and Neyman & Pearson on the other.
- Though we are guided by $\alpha = 0.05$, we *never* mention it when plain language is required.

A technical issue

- In this class we will avoid one-tailed tests.
- Why? Ask what would happen if the results were strong and in the opposite direction to what was predicted (dental example).
- But when H_0 is rejected, we still draw directional conclusions.
- For example, if x is income and y is credit card debt, we test $H_0 : \beta_1 = 0$ with a two-sided t-test.
- Say $p = 0.0021$ and $\widehat{\beta}_1 = 1.27$. We say "Consumers with higher incomes tend to have more credit card debt."
- Is this justified? We'd better hope so, or all we can say is "There is a connection between income and average credit card debt."
- Then they ask: "What's the connection? Do people with lower income have more debt?"
- And you have to say "Sorry, I don't know."
- It's a good way to get fired, or at least look silly.

The technical resolution

- Decompose the two-sided test into a set of two one-sided tests with significance level $\alpha/2$, equivalent to the two-sided test.



- In practice, just look at the sign of the regression coefficient.
- Under the surface you are decomposing the two-sided test, but you never mention it.
- *Marking rule*: If the question asks for plain language and you draw a non-directional conclusion when a directional conclusion is possible, you get half marks.

Multiple regression

$$g(x_1, \ldots, x_k) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$$

- It's the equation of a hyper-plane, a k-dimensional surface in $k+1$ dimensions.
- Again, think of a sub-population at each combination of x values.
- $g(x_1, \ldots, x_k)$ is the average response at that set of values.

$g(x_1, \ldots, x_k) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$

- Hold all the x values except x_j fixed.
- That is, do it in your mind. We are studying the function $g(\mathbf{x})$.

$$
\begin{aligned}
g(\mathbf{x}) &= \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k \\
&= (\beta_0 + \sum_{i \neq j} \beta_i x_i) + \beta_j x_j \\
&= \alpha_0 + \beta_j x_j
\end{aligned}
$$

- Another straight line.
- The slope is unaffected by where you hold those other variables constant.
- The intercept is affected, but usually nobody cares.

How to talk about it

- With all other x values held constant as x_j varies, $E(y) = \alpha_0 + \beta_j x_j$.
- We talk about it as before, but say "controlling for" or "allowing for" or "taking into account" or "correcting for" the other variables.
- Controlling for parents' income, there is no evidence of a relationship between education and career success.
- Allowing for age, there is still a tendency for adults who exercise more to have lower blood pressure.
- These results are corrected for age, sex and severity of disease.
- Holding other variables constant, a student who studies one hour more per day is predicted to have a grade point average that is 0.47 higher.

Call it *model-based control*

- This is a big selling point for multiple regression of all kinds.
- To see what happens when variables are held constant at certain values, you don't literally have to hold them constant.
- Like "controlling for number of cigarettes smoked per day ..."

- It's valid provided that the model is approximately correct.
- It's risky outside the range of the data.

Correlation-causation

- In the model, the x values are literally producing y.
- For real data, this may be true, and it may not.
- A real (non-chance) connection between x and y does establish *why* the connection exists.
- People say "Correlation does not imply causation."
- By *correlation* they mean any kind of non-independence.

Examples

- Exercise and arthritis pain.
- The Mozart effect.
- Private music lessons, athletic training.
- Baldness and wearing a hat.
- Smoking and lung cancer.
- Vitamin B and spina bifida.

Solution?

- The best solution is random assignment,
- But this is not always possible.
- Be aware of the correlation-causation issue when making plain-language statements about the results of a statistical analysis.
- Watch out for going too far beyond what the data are actually telling you.

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