Tests and Confidence Intervals with R*

The Math Data (introduced in the first R lecture)

Before the beginning of the Fall term, students in a first-year Calculus class took a diagnostic test with two parts: Pre-calculus and Calculus. Their High School Calculus marks and their marks in University Calculus were also available.

```
> math = read.table("http://www.utstat.toronto.edu/~brunner/data/legal/mathtest.txt")
> colnames(math) = c("ID","HScalc","PreCalcScore","CalcScore","UnivCalc")
> head(math)
  ID HScalc PreCalcScore CalcScore UnivCalc
1  1     65            2         0       39
2  2     54            6         2       57
3  3     77            4         4       62
4  4     80            5         2       76
5  5     87            4         4       86
6  6     53            3         1       60

> attach(math) # Make variable names available
> # PreCalc score is out of 9 and Calc score is out of 11. Convert to percentages.
> PreCalcScore = 100 * PreCalcScore/9
> CalcScore = 100 * CalcScore/11

> ###########  Fit the full regression model ##########
> fullmodel = lm(UnivCalc ~ HScalc+PreCalcScore+CalcScore)
> sumfull = summary(fullmodel); sumfull

Call:
lm(formula = UnivCalc ~ HScalc + PreCalcScore + CalcScore)

Residuals:
    Min      1Q  Median      3Q     Max 
-48.699  -7.954   1.603   9.242  30.260 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept)  -6.32155    6.01019  -1.052  0.29376    
HScalc        0.70097    0.08133   8.619  4.4e-16 ***
PreCalcScore  0.16849    0.05182   3.252  0.00128 ** 
CalcScore     0.08722    0.04282   2.037  0.04257 *  
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1 

Residual standard error: 14.16 on 291 degrees of freedom
Multiple R-squared:  0.3583,  Adjusted R-squared:  0.3517 
F-statistic: 54.17 on 3 and 291 DF,  p-value: < 2.2e-16
```

Question: What if we treated the t-tests as follow-ups to the overall F-test of H_0: $\beta_1 = \beta_2 = \beta_3 = 0$?

```
> 0.00128*3
[1] 0.00384
```

* Copyright information is on the last page.

$$t = \frac{\mathbf{a}'\widehat{\beta} - \mathbf{a}'\beta}{\sqrt{MSE\,\mathbf{a}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{a}}} \sim t(n-k-1)$$

```
> # Reproduce the test for H0: beta1=0
> a = as.matrix(c(0,1,0,0)); a
     [,1]
[1,]    0
[2,]    1
[3,]    0
[4,]    0
> betahat = coefficients(fullmodel)
> MSE.XpXinv = vcov(fullmodel) # Very handy
> dfe = df.residual(fullmodel) # n-k-1
> tstat = t(a) %*% betahat / sqrt( t(a)%*%MSE.XpXinv%*%a ) # tstat is a 1x1 matrix
> tstat = as.numeric(tstat) # Make it a number
> pvalue = 2 * ( 1-pt(abs(tstat),dfe) ) # 2-sided p-value
> c(tstat,pvalue) # Compare t = 8.619, p = 4.4e-16
[1] 8.618832e+00 4.440892e-16

> # Where do the "standard errors" come from?
> sumfull$coefficients
                Estimate Std. Error   t value     Pr(>|t|)
(Intercept)  -6.32155443 6.01018690 -1.051807 2.937609e-01
HScalc        0.70097203 0.08133028  8.618832 4.400339e-16
PreCalcScore  0.16848750 0.05181513  3.251704 1.282019e-03
CalcScore     0.08721742 0.04281987  2.036845 4.257051e-02
> sqrt(diag(MSE.XpXinv))
 (Intercept)       HScalc PreCalcScore    CalcScore 
  6.01018690   0.08133028   0.05181513   0.04281987 
```

```
> ########### Confidence intervals ###########
```

$$\mathbf{a}'\widehat{\beta} \pm t_{\alpha/2}\sqrt{MSE\,\mathbf{a}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{a}}$$

```
>
> ########### Confidence intervals ###########
> # Confidence interval for beta_j is just betahat_j plus or minus t_{alpha/2} * SE
> t.025 = qt(0.975,dfe); t.025
[1] 1.96815
> # If High School calculus is 10 marks higher, expected university calculus increases by
> # 10*beta1. Estimate the increase and give a 95% confidence interval.
> # 0.95 = P(A < beta1 < B) = P(10*A < 10*beta1 < 10*B)
> me95 = 10 * t.025*SE[2]
> cat("University calculus mark is expected to be",10*betahat[2],"points higher, plus or
minus",me95,"\n")
University calculus mark is expected to be 7.00972 points higher, plus or minus 1.600702
```

```
> # Confidence interval for beta3-beta2
> a = as.matrix(c(0,0,-1,1)) # Now a is different
> se = sqrt( t(a)%*%MSE.XpXinv%*%a ) # Standard error of the difference
> me95 = as.numeric( t.025*se ) # Now me95 is different
> estdiff = as.numeric( t(a) %*% betahat ); estdiff
[1] -0.08127008
> Lower95 = estdiff - me95; Upper95 = estdiff + me95
> c(Lower95, Upper95)
[1] -0.23598624  0.07344609
```

```
> ########### General linear test ###########
```

$$F^* = \frac{(\mathbf{C}\widehat{\beta} - \mathbf{t})'(\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}')^{-1}(\mathbf{C}\widehat{\beta} - \mathbf{t})}{q\,MSE} \overset{H_0}{\sim} F(q, n-k-1)$$

```
> # Reproduce the overall test: F = 54.17
> # First do it the hard way
> C1 = rbind( c(0,1,0,0),
+             c(0,0,1,0),
+             c(0,0,0,1) )
> q = dim(C1)[1] # Number of rows
> F1 = t(C1%*%betahat-0) %*% solve(C1 %*% MSE.XpXinv %*% t(C1)) %*%
+                         (C1%*%betahat-0) / q
> F1 # It's a 1x1 matrix
         [,1]
[1,] 54.16969
```

```
> # The easy way: Use this function if you like.
> source("http://www.utstat.utoronto.ca/~brunner/Rfunctions/ftest.txt")
> ftest # See the function definition
function(model,L,h=0)
# General linear test of H0: L beta = h
    {
    BetaHat = model$coefficients
    dimL = dim(L)
    if(length(BetaHat) != dimL[2]) stop("Sizes of L and Beta are incompatible")
    r = dimL[1]
    if(qr(L)$rank != r) stop("Rows of L must be linearly independent.")
    out = numeric(4)
    names(out) = c("F","df1","df2","p-value")
    dfe = df.residual(model)
    diff = L%*%BetaHat-h
    fstat = t(diff) %*% solve(L%*%vcov(model)%*%t(L)) %*% diff / r
    # Note vcov = MSE * XtXinv
    fstat = as.numeric(fstat)
    out[1] = fstat; out[2]=r; out[3]=dfe
    out[4] = 1-pf(fstat,r,dfe)
    return(out)
    }
> ftest(fullmodel,C1) # Compare F = 54.17
        F       df1       df2   p-value
 54.16969   3.00000 291.00000   0.00000

> # Test the two subtests simultaneously, controlling for HS Calculus mark
> C2 = rbind( c(0,0,1,0),
+             c(0,0,0,1) )
> ftest(fullmodel,C2)
           F          df1          df2      p-value
1.144444e+01 2.000000e+00 2.910000e+02 1.642604e-05

> round(ftest(fullmodel,C2),4)
       F      df1      df2  p-value
 11.4444   2.0000 291.0000   0.0000

> # Do it with the full-reduced approach
```

$$F^* = \frac{SSR(full) - SSR(reduced)}{q\ MSE}$$

```
> JustHScalc = lm(UnivCalc ~ HScalc) # The reduced (restricted) model
> anova(JustHScalc,fullmodel)

Analysis of Variance Table

Model 1: UnivCalc ~ HScalc
Model 2: UnivCalc ~ HScalc + PreCalcScore + CalcScore
  Res.Df   RSS Df Sum of Sq      F    Pr(>F)
1    293 62967
2    291 58375  2    4591.5 11.444 1.643e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

The SAT Data

In the United States, admission to university is based partly on high school marks and recommendations, and partly on applicants' performance on a standardized multiple choice test called the Scholastic Aptitude Test (SAT). The SAT has two sub-tests, Verbal and Math. A university administrator selected a random sample of 200 applicants, and recorded the Verbal SAT, the Math SAT and first-year university Grade Point Average (GPA) for each student. We seek to predict GPA from the two test scores. Throughout, we will use the $\alpha = 0.05$ significance level.

1. First, fit a model using just the Math score as a predictor. “Fit” means estimate the model parameters.

```
> rm(list=ls())
> sat = read.table("http://www.utstat.toronto.edu/~brunner/data/legal/openSAT.data.txt")
> head(sat)
  VERBAL MATH  GPA
1    578  567 2.68
2    474  653 2.51
3    546  657 1.95
4    664  686 2.81
5    600  619 2.79
6    488  738 2.36
> # Q1
> justmath = lm(GPA ~ MATH, data=sat); summary(justmath)

Call:
lm(formula = GPA ~ MATH, data = sat)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.92390 -0.38854 -0.00325  0.38448  1.43043 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 1.5272240  0.3981544   3.836 0.000168 ***
MATH        0.0016979  0.0006098   2.784 0.005885 ** 
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.5707 on 198 degrees of freedom
Multiple R-squared:  0.03768, Adjusted R-squared:  0.03282 
F-statistic: 7.752 on 1 and 198 DF,  p-value: 0.005885
```

Does there appear to be a relationship between Math score and grade point average?

a. Answer Yes or No.

b. Fill in the blank. Students who did better on the Math test tended to have ________ first-year grade point average.

c. Do you reject H_0: $\beta_1 = 0$? Answer Yes or No.

d. Are the results statistically significant? Answer Yes or No.

e. What is the *p*-value? The answer can be found in *two* places on your printout.

f. What proportion of the variation in first-year grade point average is explained by score on the SAT Math test? The answer is a number from your printout.

g. Give a predicted first-year grade point average for a student who got 700 on the Math SAT. The answer is a number you could get with a calculator from the output of `summary`, but do it with R.

```
> betahat = coefficients(justmath); betahat
(Intercept)        MATH 
1.527224026 0.001697934 
> betahat[1]+700*betahat[2]
(Intercept) 
   2.715778 
```

2. Now fit a model with both the Math and Verbal sub-tests.

```
> 
> # Q2
> 
> fullmodel = lm(GPA ~ VERBAL + MATH, data=sat); summary(fullmodel)

Call:
lm(formula = GPA ~ VERBAL + MATH, data = sat)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.70296 -0.36750  0.02644  0.38869  1.24830 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 0.6080747  0.4413074   1.378    0.170    
VERBAL      0.0023070  0.0005521   4.178 4.41e-05 ***
MATH        0.0009974  0.0006095   1.636    0.103    
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1 

Residual standard error: 0.5484 on 197 degrees of freedom
Multiple R-squared:  0.116,	Adjusted R-squared:  0.107 
F-statistic: 12.93 on 2 and 197 DF,  p-value: 5.305e-06 
```

a. Give the test statistic, the degrees of freedom and the *p*-value for each of the following null hypotheses. The answers are numbers from your printout.

i) H_0: $\beta_1 = \beta_2 = 0$
ii) H_0: $\beta_1 = 0$
iii) H_0: $\beta_2 = 0$
iv) H_0: $\beta_0 = 0$

b. Controlling for Verbal score, is Math score related to first-year grade point average?

i) Give the value of the test statistic. The answer is a number from your printout.
ii) Give the *p*-value. The answer is a number from your printout.
iii) Do you reject the null hypothesis? Answer Yes or No.
iv) Are the results statistically significant? Answer Yes or No.
v) In plain, non-statistical language, what do you conclude? The answer is something about test scores and grade point average.

Repeating ...

```
> # Q2
>
> fullmodel = lm(GPA ~ VERBAL + MATH, data=sat); summary(fullmodel)

Call:
lm(formula = GPA ~ VERBAL + MATH, data = sat)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.70296 -0.36750  0.02644  0.38869  1.24830 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 0.6080747  0.4413074   1.378    0.170    
VERBAL      0.0023070  0.0005521   4.178 4.41e-05 ***
MATH        0.0009974  0.0006095   1.636    0.103    
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1 

Residual standard error: 0.5484 on 197 degrees of freedom
Multiple R-squared:  0.116,	Adjusted R-squared:  0.107 
F-statistic: 12.93 on 2 and 197 DF,  p-value: 5.305e-06
```

c. Allowing for Math score, is Verbal score related to first-year grade point average?

i) Give the value of the test statistic. The answer is a number from your printout.

ii) Give the *p*-value. The answer is a number from your printout.

iii) Do you reject the null hypothesis? Answer Yes or No.

iv) Are the results statistically significant? Answer Yes or No.

v) In plain, non-statistical language, what do you conclude? The answer is something about test scores and grade point average.

vi) Do the test with the full-reduced model approach. Does $F = t^2$? Compare the *p*-values.

```
> anova(justmath,fullmodel)
Analysis of Variance Table

Model 1: GPA ~ MATH
Model 2: GPA ~ VERBAL + MATH
  Res.Df    RSS Df Sum of Sq      F    Pr(>F)    
1    198 64.486                                  
2    197 59.236  1    5.2496 17.458 4.411e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1 
```

d. Give a predicted first-year grade point average for a student who got 650 on the Verbal and 700 on the Math SAT.

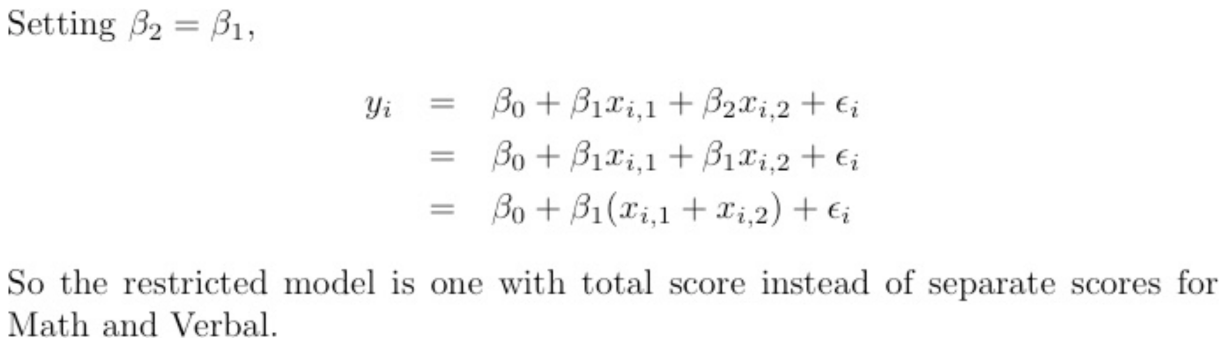
```
> betahat = coefficients(fullmodel); betahat
 (Intercept)       VERBAL         MATH 
0.6080747411 0.0023070007 0.0009973607 
> betahat[1] + 650*betahat[2] + 700*betahat[3]
(Intercept) 
   2.805778 
```

e. Let's do one more test. We want to know whether expected GPA increases faster as a function of the Verbal SAT, or the Math SAT. That is, we want to compare the regression coefficients, by testing
H_0: $\beta_1 = \beta_2$.

i) Do it with the general linear test. Feel free to use my `ftest` function.

```
> source("http://www.utstat.toronto.edu/~brunner/Rfunctions/ftest.txt")
> # To test H0: C beta = t. Input us model, C matrix, optional t (default zero)
> C = cbind(0,1,-1); C # One row
     [,1] [,2] [,3]
[1,]    0    1   -1
> ftest(fullmodel,C)
          F         df1         df2     p-value
  1.9909873   1.0000000 197.0000000   0.1598149
```

ii) Do the same thing with the full-reduced model approach. Show your work.



Setting $\beta_2 = \beta_1$,

$$\begin{aligned} y_i &= \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \epsilon_i \\ &= \beta_0 + \beta_1 x_{i,1} + \beta_1 x_{i,2} + \epsilon_i \\ &= \beta_0 + \beta_1 (x_{i,1} + x_{i,2}) + \epsilon_i \end{aligned}$$

So the restricted model is one with total score instead of separate scores for Math and Verbal.

```
> total = sat$VERBAL + sat$MATH
> sat = cbind(sat,total)
> reducedmodel = lm(GPA ~ total, data=sat)
>
> anova(reducedmodel,fullmodel)

Analysis of Variance Table

Model 1: GPA ~ total
Model 2: GPA ~ VERBAL + MATH
  Res.Df    RSS Df Sum of Sq     F Pr(>F)
1    198 59.835
2    197 59.236  1   0.59867 1.991 0.1598
```

iii) State your conclusion in plain, non-technical language. It's something about first-year grade point average.