STA 302 Assignment 2

1

(2.6a) $AB = \begin{pmatrix} 8 & 3 & 7 \\ -2 & 5 & -3 \end{pmatrix} \begin{pmatrix} -2 & 5 \\ 3 & 7 \\ 6 & -4 \end{pmatrix}$

$$= \begin{pmatrix} -16+9+42 & 40+21-28 \\ 4+15-18 & -10+35+12 \end{pmatrix} = \begin{pmatrix} 35 & 33 \\ 1 & 37 \end{pmatrix}$$

$BA = \begin{pmatrix} -2 & 5 \\ 3 & 7 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} 8 & 3 & 7 \\ -2 & 5 & -3 \end{pmatrix}$

$$= \begin{pmatrix} -16-10 & -6+25 & -14-15 \\ 24-14 & 9+35 & 21-21 \\ 48+8 & 18-20 & 42+12 \end{pmatrix} = \begin{pmatrix} -26 & 19 & -29 \\ 10 & 44 & 0 \\ 56 & -2 & 54 \end{pmatrix}$$

(2.6d) $tr(AB) = 35+37 = 72$

$tr(BA) = -26+44+54 = 72$

Q1 continued

(2.7b) Try $x = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, $Ax = \begin{pmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 12 & 8 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(2.7c) Both matrices have only one linearly independent row. Rank of both matrices is one. This could be verified by row reduction.

(2.17c) $\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = 2\cdot 3 - 5\cdot 1 = 1$

(2.17d) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}^{-1} = \dfrac{1}{2\cdot 3 - 5\cdot 1}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

(2.20) $2\begin{pmatrix} 5 \\ 7 \end{pmatrix} + 4\begin{pmatrix} -2 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix} + \begin{pmatrix} -8 \\ 12 \end{pmatrix} + \begin{pmatrix} -9 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} -7 \\ 23 \end{pmatrix}$, while

$Ab = \begin{pmatrix} 5 & -2 & 3 \\ 7 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 10-8-9 \\ 14+12-3 \end{pmatrix} = \begin{pmatrix} -7 \\ 23 \end{pmatrix}$

Q1 continued

(2.23) Bind the vectors into the columns of A. One column is all zeros, say the first one. Let

$$\underset{\sim}{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ and } Ax = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nk} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

satisfying the definition of linear dependence.

(2.24) If A is nonsingular (A^{-1} exists), then

$$AB = O \implies A^{-1}AB = A^{-1}O \implies B = O$$

If B^{-1} exists,

$$AB = O \implies ABB^{-1} = OB^{-1} \implies A = O$$

Since O is singular, we rule out the possibility that both are non-singular. The remaining possibility is that both matrices are singular, as in the example of (2.43)

2) If both B & C are inverses of A,

$$AB = AC = I$$

$$\implies \underbrace{CA}_{I}B = C\underbrace{AC}_{I}$$

$$\implies B = C \quad \boxtimes$$

3

(a) $1-\alpha = P\left\{-t_{1-\alpha/2} < \dfrac{\sqrt{n}(\bar{y}-\mu)}{s} < t_{1-\alpha/2}\right\}$

$= P\left\{-t_{1-\frac{\alpha}{2}} \dfrac{s}{\sqrt{n}} < \bar{y}-\mu < t_{1-\alpha/2} \dfrac{s}{\sqrt{n}}\right\}$

$= P\left\{-\bar{y}-t_{1-\alpha/2} \dfrac{s}{\sqrt{n}} < -\mu < -\bar{y}+t_{1-\alpha/2} \dfrac{s}{\sqrt{n}}\right\}$

$= P\left\{\bar{y}+t_{1-\alpha/2} \dfrac{s}{\sqrt{n}} > \mu > \bar{y}-t_{1-\alpha/2} \dfrac{s}{\sqrt{n}}\right\}$

$= P\left\{\bar{y}-t_{1-\alpha/2} \dfrac{s}{\sqrt{n}} < \mu < \bar{y}+t_{1-\alpha/2} \dfrac{s}{\sqrt{n}}\right\}$

That's $\bar{y} \pm t_{1-\alpha/2} \dfrac{s}{\sqrt{n}}$

(b) i. qt(0.975, 22) = 2.074

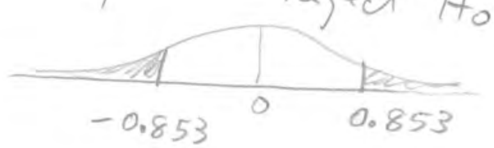
ii. $\bar{y} \pm t_{1-\alpha/2} \dfrac{s}{\sqrt{n}} = 2.57 \pm 2.074 \dfrac{\sqrt{5.85}}{\sqrt{23}}$

$= 2.57 \pm 1.046 = (1.524, 3.616)$

(c) i. $t = \dfrac{\sqrt{n}(\bar{y}-\mu_0)}{s} = \dfrac{\sqrt{23}(2.57-3)}{\sqrt{5.85}} = -0.853$

ii. $t_{1-\alpha/2} = t_{0.975} = 2.074$

iii. No, don't reject H_0.

iv. 

$2 * pt(-0.853, df=22) = 0.403$

v. No, can't conclude μ is different from 3.

vi. The answer is No. Don't be trapped by this kind of question. It is a test of your confidence.

(4) (a) $M_{ax}(t) = E\left(e^{(ax)t}\right) = E\left(e^{x(at)}\right)$

$= M_x(at)$

(b) $M_{x+a}(t) = E\left(e^{(x+a)t}\right) = E\left(e^{xt+at}\right)$

$= E\left(e^{xt}e^{at}\right) = e^{at}E\left(e^{xt}\right) = e^{at}M_x(t)$

(c) $M_{\Sigma x_i}(t) = E\left(e^{\left(\sum_{i=1}^{n} x_i\right)t}\right) = E\left(\prod_{i=1}^{n} e^{x_i t}\right)$

(Because $x_1, \dots, x_n$ are independent, the random variables $e^{x_1 t}, e^{x_2 t}, \dots, e^{x_n t}$ are also independent.)

$\overset{\text{ind}}{=} \prod_{i=1}^{n} E\left(e^{x_i t}\right) = \prod_{i=1}^{n} M_{x_i}(t)$

5 (a) $M_y(t) = M_{ax+b}(t) \underset{\text{By 4a \& 4b}}{=} e^{bt} M_x(at)$

$$= e^{bt} e^{\mu(at) + \frac{1}{2}\sigma^2 (at)^2}$$

$$= e^{(a\mu+b)t + \frac{1}{2}(a^2\sigma^2)t^2}$$

MGF of $N(a\mu+b, a^2\sigma^2)$

(b) This is (5a) with $a = \frac{1}{\sigma}$ & $b = -\frac{\mu}{\sigma}$, so

$$z \sim N\left(\frac{1}{\sigma}\mu + \left(\frac{-\mu}{\sigma}\right), \left(\frac{1}{\sigma}\right)^2 \sigma^2\right)$$

$= N(0,1)$ standard normal

(c) By (4c) $M_y(t) = \prod_{i=1}^{n} M_{x_i}(t)$

$$= \prod_{i=1}^{n} e^{\mu t + \frac{1}{2}\sigma^2 t^2} = \left(e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right)^n$$

$$= e^{(n\mu)t + \frac{1}{2}(n\sigma^2)t^2}$$

MGF of $N(n\mu, n\sigma^2)$

(5d) $M_{\bar{x}}(t) = M_{\frac{1}{n}\sum_{i=1}^{n} x_i}(t) = M_{\sum_{i=1}^{n} x_i}\left(\frac{t}{n}\right)$

$$= \prod_{i=1}^{n} M_{x_i}\left(\frac{t}{n}\right) = \prod_{i=1}^{n} e^{\mu\frac{t}{n} + \frac{1}{2}\sigma^2\left(\frac{t}{n}\right)^2}$$

$$= \left(e^{\mu\frac{t}{n} + \frac{1}{2}\sigma^2\frac{t^2}{n^2}}\right)^n = e^{\mu t + \frac{1}{2}\frac{\sigma^2}{n}t^2}$$

MGF of $N\left(\mu, \frac{\sigma^2}{n}\right)$

(e) $z = \dfrac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$ by (5b)

(f) $M_y(t) = M_{a_0 + \sum_{i=1}^{n} a_i x_i}(t) \overset{(4b)}{=} e^{a_0 t} M_{\sum a_i x_i}(t)$

$$\overset{(4c)}{=} e^{a_0 t}\prod_{i=1}^{n} M_{a_i x_i}(t) \overset{(4a)}{=} e^{a_0 t}\prod_{i=1}^{n} M_{x_i}(a_i t)$$

$$= e^{a_0 t}\prod_{i=1}^{n} e^{\mu_i(a_i t) + \frac{1}{2}\sigma_i^2(a_i t)^2}$$

$$= e^{a_0 t} e^{\sum_{i=1}^{n}\left[a_i\mu_i t + \frac{1}{2}a_i^2\sigma_i^2 t^2\right]}$$

$$= e^{\left(a_0 + \sum_{i=1}^{n} a_i\mu_i\right)t + \frac{1}{2}\left(\sum_{i=1}^{n} a_i^2\sigma_i^2\right)t^2}$$

MGF of $N\left(a_0 + \sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$

(6)

(a) $M_y(t) = M_{\Sigma x_i}(t) \overset{\text{ind}}{=} \prod_{i=1}^{n} M_{x_i}(t)$

$= \prod_{i=1}^{n} (1-2t)^{-r_i/2} = (1-2t)^{-\sum_{i=1}^{n} r_i/2}$

MGF of $\chi^2(\sum_{i=1}^{n} r_i)$

(b) $M_{z^2}(t) = E(e^{z^2 t})$

$= \int_{-\infty}^{\infty} e^{z^2 t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2(1-2t)} dz$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-2t)^{-1}} z^2} dz$

Looks like the density of a $N(0, (1-2t)^{-1})$ as long as $t < \frac{1}{2}$ so the "variance" is positive.

$= (1-2t)^{-\frac{1}{2}} \underbrace{\frac{1}{(1-2t)^{-1/2}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2(1-2t)^{-1}}} dz}_{=1}$

MGF of $\chi^2(r=1)$

(6c) $y = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2$. Each $z_i = \frac{x_i - \mu}{\sigma} \sim N(0,1)$ by (5b). Each $z_i^2 \sim \chi^2(1)$ by (6b)

The z_i^2 are independent because the x_i are independent, so $y \sim \chi^2(n)$ by (6a)

(d) By independence, $M_y(t) = M_{x_1}(t) M_{x_2}(t)$

$$\Rightarrow (1-2t)^{-(r_1+r_2)/2} = M_{x_1}(t)\,(1-2t)^{-r_2/2}$$

$$\Rightarrow (1-2t)^{-r_1/2} \cancel{(1-2t)^{-r_2/2}} = M_{x_1}(t) \cancel{(1-2t)^{-r_2/2}}$$

$$\Rightarrow M_{x_1}(t) = (1-2t)^{-r_1/2}$$

MGF of $\chi^2(r_1)$

(6e) Following the hint,

$$\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}(x_i-\bar{x}+\bar{x}-\mu)^2$$

$$= \sum_{i=1}^{n}\left[(x_i-\bar{x})^2 + 2(x_i-\bar{x})(\bar{x}-\mu) + (\bar{x}-\mu)^2\right]$$

$$= \sum_{i=1}^{n}(x_i-\bar{x})^2 + 2(\bar{x}-\mu)\underbrace{\sum_{i=1}^{n}(x_i-\bar{x})}_{=0} + n(\bar{x}-\mu)^2$$

$$= \sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2$$

$$\Longrightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2}$$

$$\Longrightarrow \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 = \frac{(n-1)s^2}{\sigma^2} + \left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2$$

$$y \quad = \quad x_1 \quad + \quad x_2 \quad \text{as in (6d)}$$

- $y \sim \chi^2(n)$ by (6c)
- $x_2 \sim \chi^2(1)$ by (5e) and (6b)
- x_1 & x_2 are independent because $\bar{x}$ & s^2 are independent

Therefore by (6d), $x_1 = \dfrac{(n-1)s^2}{\sigma^2}$ is χ^2, with df $r_1 = n-1$

7 (a) $Q(\beta_0, \beta_1) = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$

$$\frac{\partial Q}{\partial \beta_0} = \sum_{i=1}^{n} 2(y_i - \beta_0 - \beta_1 x_i)(-1) \overset{set}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} y_i = n\beta_0 + \beta_1 \sum_{i=1}^{n} x_i$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}(n\beta_0 + \beta_1 \Sigma x_i) = \beta_0 + \beta_1 \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\Rightarrow \beta_0 = \bar{y} - \beta_1 \bar{x} \qquad (1)$$

$$\frac{\partial Q}{\partial \beta_1} = \sum_{i=1}^{n} 2(y_i - \beta_0 - \beta_1 x_i)(-x_i)$$

$$= -2\sum_{i=1}^{n} (x_i y_i - \beta_0 x_i - \beta_1 x_i^2)$$

$$= -2\left(\sum_{i=1}^{n} x_i y_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2\right) \overset{set}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i = n\bar{x}\beta_0 + \beta_1 \sum_{i=1}^{n} x_i^2 \qquad (2)$$

Substituting (1) into (2)

Next page

(7a continued)

$$\sum_{i=1}^{n} x_i y_i = n\bar{x}\Big(\overbrace{\bar{y}-\beta_1\bar{x}}^{\beta_0}\Big) + \beta_1 \sum_{i=1}^{n} x_i^2$$

$$= n\bar{x}\bar{y} - n\bar{x}^2\beta_1 + \beta_1\sum_{i=1}^{n} x_i^2$$

$$= n\bar{x}\bar{y} + \beta_1\Big(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\Big)$$

$$\Longrightarrow \beta_1\Big(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\Big) = \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}$$

$$\Longrightarrow \beta_1 = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

Putting hats on to indicate that these values of β_0 and β_1 where the minimum of Q occurs are <u>estimates</u>,

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

Problems 11b & 12 from Assignment 1

and

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$$

(7b) $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \sim N(\beta_0 + \beta_1 x_i \,,\, \sigma^2)$

(c) $$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i - \sum_{i=1}^{n}(x_i-\bar{x})\bar{y}}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i}{\sum_{i=1}^{n}(x_i-\bar{x})^2} - \frac{\bar{y}\sum_{i=1}^{n}(x_i-\bar{x})}{\sum_{i=1}^{n}(x_i-\bar{x})^2} \quad =0$$

(d) i. By (5f), linear combination of normals is normal, so we only need to calculate the expected value and variance

$$E(\hat{\beta}_1) = \frac{\sum_{i=1}^{n}(x_i-\bar{x})E(y_i)}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(\beta_0+\beta_1 x_i)}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$= \frac{\beta_0 \overbrace{\sum_{i=1}^{n}(x_i-\bar{x})}^{=0} + \beta_1 \sum_{i=1}^{n}(x_i-\bar{x})x_i}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \frac{0 + \beta_1 \sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$= \beta_1 \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \beta_1$$

So $\hat{\beta}_1$ is unbiased

(7d i continued)

$$\text{Var}(\hat{\beta}_1) = \text{Var}\left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right) = \text{Var}\left(\sum_{i=1}^{n}\frac{(x_i-\bar{x})}{\sum_{j=1}^{n}(x_j-\bar{x})^2}\,y_i\right)$$

$$\overset{\text{ind}}{\downarrow}\!\!= \sum_{i=1}^{n}\text{Var}\left(\frac{(x_i-\bar{x})}{\sum_{j=1}^{n}(x_j-\bar{x})^2}\,y_i\right) = \sum_{i=1}^{n}\left(\frac{x_i-\bar{x}}{\sum_{j=1}^{n}(x_j-\bar{x})^2}\right)^2 \text{var}(y_i)$$

$$= \sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{\left[\sum_{j=1}^{n}(x_j-\bar{x})^2\right]^2}\sigma^2 = \sigma^2\,\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\left[\sum_{j=1}^{n}(x_j-\bar{x})^2\right]^2}$$

$$= \frac{\sigma^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\ , \text{ so}$$

$$\hat{\beta}_1 \sim N\left(\beta_1\ ,\ \frac{\sigma^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)$$

(7d ii) Extending Questions (9a) & (10) of Assignment One (see formula sheet)

$$\text{Cov}(\bar{y}, \hat{\beta}_1) = \text{Cov}\left(\frac{1}{n}\sum_{i=1}^{n} y_i, \frac{1}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\sum_{i=1}^{n}(x_i-\bar{x})y_i\right)$$

$$= \frac{1}{n\sum_{i=1}^{n}(x_i-\bar{x})^2}\text{Cov}\left(\sum_{i=1}^{n} y_i, \sum_{j=1}^{n}(x_j-\bar{x})y_j\right)$$

$$= \frac{1}{n\sum_{i=1}^{n}(x_i-\bar{x})^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\text{Cov}(y_i, (x_j-\bar{x})y_j)$$

$$= \frac{1}{n\sum_{i=1}^{n}(x_i-\bar{x})^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_j-\bar{x})\underbrace{\text{Cov}(y_i, y_j)}$$

By independence, $= 0$ when $i \neq j$

$$= \frac{1}{n\sum_{i=1}^{n}(x_i-\bar{x})^2}\sum_{i=1}^{n}(x_i-\bar{x})\text{Cov}(y_i, y_i)$$

$$= \frac{1}{n\sum_{i=1}^{n}(x_i-\bar{x})^2}\sum_{i=1}^{n}(x_i-\bar{x})\sigma^2$$

$$= \frac{\sigma^2}{n\sum_{i=1}^{n}(x_i-\bar{x})^2}\underbrace{\sum_{i=1}^{n}(x_i-\bar{x})}_{=0}$$

$$= 0$$

(7d iii) Again, linear combination of normals is normal. Only need to calculate the expected value and variance

$$E(\hat{\beta}_0) = E(\bar{y} - \hat{\beta}_1 \bar{x}) = E(\bar{y}) - \bar{x}E(\hat{\beta}_1)$$

$$= E\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) - \bar{x}\beta_1 = \frac{1}{n}\sum_{i=1}^{n} E(y_i) - \beta_1 \bar{x}$$

$$= \frac{1}{n}\sum_{i=1}^{n}(\beta_0 + \beta_1 x_i) - \beta_1 \bar{x}$$

$$= \frac{1}{n} n\beta_0 + \beta_1 \frac{1}{n}\sum_{i=1}^{n} x_i - \beta_1 \bar{x} = \beta_0 + \beta_1\bar{x} - \beta_1 \bar{x}$$

$$= \beta_0$$

$$Var(\hat{\beta}_0) = Var(\bar{y} - \bar{x}\hat{\beta}_1) =$$

$$= Var(\bar{y}) + \bar{x}^2 Var(\hat{\beta}_1) - 2\bar{x}\,Cov(\bar{y}, \hat{\beta}_1)$$

$$= Var\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) + \bar{x}^2 \frac{\sigma^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} - 0 \leftarrow (7d\,ii)$$

$$\overset{\text{ind}}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(y_i) + \frac{\bar{x}^2\sigma^2}{\sum(x_i-\bar{x})^2}$$

$$= \frac{1}{n^2} n\sigma^2 + \frac{\bar{x}^2\sigma^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \frac{\sigma^2}{n} + \frac{\bar{x}^2\sigma^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$= \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right) = \sigma^2\left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2 + n\bar{x}^2}{n\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)$$

$$= \sigma^2 \frac{\sum x_i^2 - \cancel{n\bar{x}^2} + \cancel{n\bar{x}^2}}{n\sum_{i=1}^{n}(x_i-\bar{x})^2}, \text{ so } \hat{\beta}_0 \sim N\left(\beta_0, \frac{\sigma^2 \sum x_i^2}{\sum(x_i-\bar{x})^2}\right)$$

(7d iv) $\mathrm{Cov}(\hat\beta_0, \hat\beta_1) = \mathrm{Cov}(\bar y - \hat\beta_1 \bar x, \hat\beta_1)$

$= \mathrm{Cov}(\bar y, \hat\beta_1) - \bar x \,\mathrm{Cov}(\hat\beta_1, \hat\beta_1)$

$= 0 - \bar x \,\mathrm{Var}(\hat\beta_1) = \dfrac{-\bar x \sigma^2}{\sum_{i=1}^{n} (x_i - \bar x)^2}$

```
> # 7e
> x = c( 0.0,  1.3,  3.2, -2.5, -4.6, -1.6,  4.5,  3.8)
> y = c(-0.8, -1.3,  7.4, -5.2, -6.5, -4.9,  9.9,  7.2)

> ybar = mean(y); xbar = mean(x); ss = sum((x-xbar)^2)
> beta1hat = sum((x-xbar)*(y-ybar))/ss; beta0hat = ybar - beta1hat*xbar

> c(beta0hat,beta1hat)
[1] -0.2497055  1.9018644

> lm(y~x)  # Check

Call:
lm(formula = y ~ x)

Coefficients:
(Intercept)            x
    -0.2497       1.9019
```