Assignment 6

(1) $y = (w-\mu)'\Sigma^{-1}(w-\mu) = z'z$, where
$z = \Sigma^{-\frac{1}{2}}(w-\mu)$. z is multivariate normal
with $E(z)=0$ and $\text{cov}(z) = \Sigma^{-\frac{1}{2}}\Sigma\Sigma^{-\frac{1}{2}'}$

$$= I$$

So $z \sim N_p(0, I)$ and the elements of z are independent standard normals. Then

$$z'z = \sum_{j=1}^{p} z_j^2 \sim \chi^2(p), \text{ sum of } \chi^2 \text{ is } \chi^2 \quad \blacksquare$$

(2) (a) $\text{Cov}(\bar{y}, y_j - \bar{y}) = E(\bar{y}(y_j - \bar{y})) - E(\bar{y})\underbrace{E(y_j - \bar{y})}_{0}$

$$= E(\bar{y}(y_j - \bar{y})) - 0$$

$$= E(y_j \bar{y}) - E(\bar{y}^2) = E\left(y_j \frac{1}{n}\sum_{i=1}^{n} y_i\right) - \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(y_i y_j) - \frac{\sigma^2}{n} - \mu^2$$

$$= \frac{1}{n}\left(E(y_j^2) + \sum_{i \neq j}^{n-1} E(y_i)E(y_j)\right) - \frac{\sigma^2}{n} - \mu^2$$

$$= \frac{1}{n}\left(\sigma^2 + \mu^2 + (n-1)\mu^2\right) - \frac{\sigma^2}{n} - \mu^2$$

$$= \frac{\sigma^2}{n} + \mu^2 - \frac{\sigma^2}{n} - \mu^2 = 0$$

(2b) Because zero covariance implies independence for the multivariate normal, $\bar{y}$ is independent of $\begin{pmatrix} y_1 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix}$. Since functions of independent random variables are independent, this makes $\bar{y}$ independent of $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$

(c) $\sum_{i=1}^{n} (y_i - \mu)^2 = \sum_{i=1}^{n} (y_i - \bar{y} + \bar{y} - \mu)^2$

$$= \sum_{i=1}^{n} \left((y_i - \bar{y})^2 + 2(\bar{y} - \mu)(y_i - \bar{y}) + (\bar{y} - \mu)^2 \right)$$

$$= \sum_{i=1}^{n} (y_i - \bar{y})^2 + 2(\bar{y} - \mu) \underbrace{\sum_{i=1}^{n} (y_i - \bar{y})}_{0} + n(\bar{y} - \mu)^2$$

$= \sum_{i=1}^{n} (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2$, so that

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \mu)^2 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{\sigma^2} + \frac{n(\bar{y} - \mu)^2}{\sigma^2}$$

$$\Rightarrow \sum_{i=1}^{n} \left(\frac{y_i - \mu}{\sigma} \right)^2 = \frac{(n-1)s^2}{\sigma^2} + \left(\frac{\bar{y} - \mu}{\sigma / \sqrt{n}} \right)^2$$

$$W \quad = \quad W_1 \quad + \quad W_2$$

3c cont.

$W \sim \chi^2(n)$ because

- $\frac{y_i - \mu}{\sigma} \sim N(0,1)$
- Square of standard normal is $\chi^2(1)$
- $\left(\frac{y_i - \mu}{\sigma}\right)^2$ are independent because y_i are independent
- Sum of independent χ^2s is χ^2, df = sum of df

$W_2 \sim \chi^2(1)$ because

- $\bar{y} \sim N(\mu, \frac{\sigma^2}{n})$, so
- $\frac{\bar{y} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$
- Square of standard normal is $\chi^2(1)$

W_1 & W_2 are independent because W_1 is a function of s^2, W_2 is a function of $\bar{y}$, & s^2 and $\bar{y}$ are independent

Finally, $W_1 \sim \chi^2(n-1)$ follows from the 4th line on the formula sheet. I need to number these or something.

(4) Using $\text{cov}(A\mathbf{y}, B\mathbf{y}) = A\,\text{cov}(\mathbf{y})\,B'$,

$$\text{cov}(b, e) = \text{cov}\left((X'X)^{-1}X'\mathbf{y},\ (I-H)\mathbf{y}\right)$$

$$= (X'X)^{-1}X'\sigma^2 I_n (I-H)'$$

$$= \sigma^2 (X'X)^{-1}X'(I-H)$$

$$= \sigma^2\left((X'X)^{-1}X' - \underbrace{(X'X)^{-1}X'X}_{I}(X'X)^{-1}X'\right)$$

$$= \sigma^2\left((X'X)^{-1}X' - (X'X)^{-1}X'\right) = 0$$

Since zero covariance implies independence for the multivariate normal, b & e are independent. This also makes b independent of $e'e$, a function of e.

(5) $\text{cov}(e, \hat{\mathbf{y}}) = \text{cov}((I-H)\mathbf{y},\ H\mathbf{y})$

$$= (I-H)\sigma^2 I H' = \sigma^2(H - HH)$$

$$= \sigma^2(H - H) = 0_{n\times n}$$

With the assumption of normality, e is independent of b & hence of $\hat{\mathbf{y}} = Xb$, and independence implies zero covariance.

(6) The formula sheet says if $w \sim N_p(\mu, \Sigma)$ with Σ positive definite, $(w-\mu)'\Sigma^{-1}(w-\mu) \sim \chi^2(p)$. Call this $(*)$.

$$\frac{1}{\sigma^2}(y - X\beta)'(y - X\beta) = \frac{e'e}{\sigma^2} + \frac{1}{\sigma^2}(b-\beta)'X'X(b-\beta)$$

$$\Rightarrow (y - X\beta)'(\sigma^2 I_n)^{-1}(y - X\beta) = \frac{e'e}{\sigma^2} + (b-\beta)'(\sigma^2(X'X)^{-1})^{-1}(b-\beta)$$

$$\begin{array}{ccccc} \| & & \| & & \| \\ W & = & W_1 & + & W_2 \end{array}$$

Because $y \sim N(X\beta, \sigma^2 I_n)$, $(*)$ says $W \sim \chi^2(n)$

Because $b \sim N(\beta, \sigma^2(X'X)^{-1})$, $(*)$ says

$W_2 \sim \chi^2(k+1)$

W_1 & W_2 are independent because W_1 is a function of e & W_2 is a function of b, and b & e are independent.

$W_1 = \frac{e'e}{\sigma^2} \sim \chi^2(n-k-1)$ follows from the 4 th line of the formula sheet, with

$\nu_1 + \nu_2 = n$ and $\nu_2 = k+1$

7 (a) $b \sim N(\beta, \sigma^2 (X'X)^{-1})$, so

$$\ell' b \sim N(\ell'\beta, \ell'\sigma^2 (X'X)^{-1}\ell)$$
$$= N(\ell'\beta, \sigma^2 \ell'(X'X)^{-1}\ell)$$

(b)
$$z = \frac{\ell' b - \ell'\beta}{\sqrt{\sigma^2 \ell'(X'X)^{-1}\ell}}$$

(c)
$$t = \frac{z}{\sqrt{w/r}} = \frac{\dfrac{\ell' b - \ell'\beta}{\sqrt{\sigma^2 \ell'(X'X)^{-1}\ell}}}{\sqrt{\dfrac{e'e}{\sigma^2} / (n-k-1)}}$$

$$= \frac{\ell' b - \ell'\beta}{\sqrt{\ell'(X'X)^{-1}\ell}\,\sqrt{s^2}} = \frac{\ell' b - \ell'\beta}{s\sqrt{\ell'(X'X)^{-1}\ell}}$$

(d) Numerator is a function of b, denominator is a function of e, and b & e are independent.

(e)
$$t = \frac{\ell' b - \gamma}{s\sqrt{\ell'(X'X)^{-1}\ell}}$$

If $k=4$

(7f) $H_0: \beta_2 = 0 \quad \ell' = (0\,0\,1\,0\,0)$

(g) $H_0: \beta_1 = \beta_2 \quad \ell' = (0\;1\;-1\;0\;0)$

(h) $\ell' = (1, 91, 83, 24)$

(i) $1-\alpha = P\{-t_{\alpha/2} < t < t_{\alpha/2}\}$

$$= P\left\{-t_{\alpha/2} < \frac{\ell' b - \ell'\beta}{s\sqrt{\ell'(X'X)^{-1}\ell}} < t_{\alpha/2}\right\}$$

$$= P\{-t_{\alpha/2}\, s\sqrt{\ell'(X'X)^{-1}\ell} < \ell' b - \ell'\beta < t_{\alpha/2}\, s\sqrt{\ell'(X'X)^{-1}\ell}\}$$

$$= P\{-\ell' b - t_{\alpha/2}\, s\sqrt{\ell'(X'X)^{-1}\ell} < -\ell'\beta < -\ell' b + t_{\alpha/2}\, s\sqrt{\ell'(X'X)^{-1}\ell}\}$$

$$= P\{\ell' b - t_{\alpha/2}\, s\sqrt{\ell'(X'X)^{-1}\ell} < \ell'\beta < \ell' b + t_{\alpha/2}\, s\sqrt{\ell'(X'X)^{-1}\ell}\},$$

or $\ell' b \pm t_{\alpha/2}\, s\sqrt{\ell'(X'X)^{-1}\ell}$

8 (a) $Cb \sim N_m(C\beta, \sigma^2 C(X'X)^{-1}C')$

(b) The formula sheet says if $w \sim N_p(\mu, \Sigma)$ then $(w-\mu)'\Sigma^{-1}(w-\mu) \sim \chi^2(p)$, so with $w = Cb$, $\mu = C\beta \overset{H_0}{=} \gamma$ and $\Sigma = \sigma^2 C(X'X)^{-1}C'$,

$$\frac{1}{\sigma^2}(Cb-\gamma)'(C(X'X)^{-1}C')^{-1}(Cb-\gamma)$$

$$= (Cb-\gamma)'(\sigma^2 C(X'X)^{-1}C')^{-1}(Cb-\gamma) \sim \chi^2(m)$$

(c) By the formula sheet $\frac{e'e}{\sigma^2} \sim \chi^2(n-k-1)$, which is independent of the chi-squared random variable in (b) because e & b are independent. Then under H_0,

$$F = \frac{w_1/r_1}{w_2/r_2} = \frac{\cancel{\frac{1}{\sigma^2}}(Cb-\gamma)'(C(X'X)^{-1}C')^{-1}(Cb-\gamma)/m}{\cancel{\frac{1}{\sigma^2}}\underbrace{e'e/(n-k-1)}_{s^2}}$$

$$= \frac{(Cb-\gamma)'(C(X'X)^{-1}C')^{-1}(Cb-\gamma)}{m s^2} \sim F(m, n-k-1)$$

1×1

(9)

$$F = \frac{(l'b-\gamma)'(l'(X'X)^{-1}l)^{-1}(l'b-\gamma)}{1 \cdot s^2}$$

$$= \frac{(l'b-\gamma)^2}{s^2\, l'(X'X)^{-1}l}$$

$$= \left(\frac{l'b-\gamma}{s\sqrt{l'(X'X)^{-1}l}} \right)^2 = t^2$$

(10)

$$F^* = \frac{(ACb-A\gamma)'(AC(X'X)^{-1}(AC)')^{-1}(ACb-A\gamma)}{m\, s^2}$$

$$= \frac{(A(Cb-\gamma))'(AC(X'X)^{-1}C'A')^{-1}A(Cb-\gamma)}{m\, s^2}$$

$$= \frac{(Cb-\gamma)'\underbrace{A'A'^{-1}}_{I}(C(X'X)^{-1}C')^{-1}\underbrace{A^{-1}A}_{I}(Cb-\gamma)}{m\, s^2}$$

$$= \frac{(Cb-\gamma)'(C(X'X)^{-1}C')^{-1}(Cb-\gamma)}{m\, s^2}$$

No change.

(11) If $\hat{y}$ is a function of b then it is independent of e, because b & e are independent, and functions of independent random vectors are independent.

(a) If $X'e = 0$, $\sum_{i=1}^{n} e_i = 0$ if the model has an intercept

(b) $0 = \sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - \hat{y}_i) = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \hat{y}_i$

so $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \hat{y}_i$

(c) $\hat{y} = Xb$, so $\frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} \hat{y}_i$ is a function of b & so it is independent of e.

(12) $SSE = \sum_{i=1}^{n} e_i^2$, and $SSR = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2$ is a function of b because $\hat{y} = Xb$ is a function of b and by problem 11c, $\bar{y}$ is also a function of b.

Since SSE is a function of e and SSR is a function of b, and b & e are independent, SSE & SSR are independent.

(13) (a) Under H_0, $y_1 \cdots y_n \overset{iid}{\sim} N(\beta_0, \sigma^2)$

(b) $SST = \dfrac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{\sigma^2} \sim \chi^2(n-1)$

(14)

$$\frac{SST}{\sigma^2} = \frac{SSE}{\sigma^2} + \frac{SSR}{\sigma^2}$$

$$w = w_1 + w_2$$

$w \sim \chi^2(n-1)$ by Problem 13 when H_0 is true

$w_1 \sim \chi^2(n-k-1)$, always

w_1 & w_2 are independent by Problem 12,

So $w_2 = \dfrac{SSR}{\sigma^2}$ is chi-squared, with df

$n-1-(n-k-1) = \cancel{n}\cancel{-1}\cancel{-n}+k\cancel{+1} = k$

(15) $\dfrac{SSR}{\sigma^2} \sim \chi^2(k)$ by Problem 14 when H_0 is true

$\dfrac{SSR}{\sigma^2}$ & $\dfrac{SSE}{\sigma^2}$ are independent by Problem 12

$\dfrac{SSE}{\sigma^2} \sim \chi^2(n-k-1)$ by the formula sheet whether H_0 is true or not. Then

$$\frac{\frac{SSR}{\sigma^2}/k}{\frac{SSE}{\sigma^2}/(n-k-1)} = \frac{SSR/k}{SSE/(n-k-1)} \overset{H_0}{\sim} F(k, n-k-1).$$

(16) $$F = \frac{SSR/k}{SSE/(n-k-1)} = \left(\frac{n-k-1}{k}\right) \frac{SSR/SST}{SSE/SST}$$

$$= \left(\frac{n-k-1}{k}\right) \frac{R^2}{(SST-SSR)/SST}$$

$$= \left(\frac{n-k-1}{k}\right) \frac{R^2}{1-R^2}$$

Taking logs and differentiating,

$$\frac{\partial}{\partial R^2}\left(\ln\left(\frac{n-k-1}{k}\right) + \ln R^2 - \ln(1-R^2) \right)$$

$$= 0 + \frac{1}{R^2} - \frac{1}{1-R^2}(-1)$$

$$= \frac{1}{R^2} + \frac{1}{1-R^2} > 0,$$ so the function is increasing