Assignment 5

(1) (a) $E(\bar{y}) = \mu$, $Var(\bar{y}) = \frac{\sigma^2}{n}$

(b) $\mu = E(L) = E\left(\sum_{i=1}^{n} c_i y_i\right) = \sum_{i=1}^{n} c_i E(y_i)$

$= \sum_{i=1}^{n} c_i \mu = \mu \sum_{i=1}^{n} c_i$

If $\mu = 0$, $c_1 \ldots c_n$ could be anything and $E(L) = \mu$

If $\mu \neq 0$, divide both sides by μ, and

$\sum_{i=1}^{n} c_i = 1$ This also works if $\mu = 0$

(c) Yes. $c_i = \frac{1}{n}$ for $i = 1, \ldots, n$

(d) $Var(L) = Var\left(\sum_{i=1}^{n} c_i y_i\right) \overset{\text{Indep.}}{=} \sum_{i=1}^{n} Var(c_i y_i)$

$= \sum_{i=1}^{n} c_i^2 \sigma^2 = \sigma^2 \sum_{i=1}^{n} c_i^2$

(e) $Var(L) - Var(\bar{y}) = \sigma^2 \sum_{i=1}^{n} c_i^2 - \frac{\sigma^2}{n}$

$= \sigma^2 \left(\sum_{i=1}^{n} c_i^2 - \frac{1}{n}\right)$

$= \sigma^2 \left(\sum_{i=1}^{n} c_i^2 - \frac{\left(\sum_{i=1}^{n} c_i\right)^2}{n}\right)$

$= \sigma^2 \sum_{i=1}^{n} (c_i - \bar{c})^2$ by the hint

≥ 0

Now if $Var(L) = Var(\bar{y})$, $\sum_{i=1}^{n} (c_i - \bar{c})^2 = 0$ and for $i = 1, \ldots, n$ $c_i = \bar{c} = \frac{\sum c_i}{n} = \frac{1}{n}$ and $L = \bar{y}$.

Thus if $L \neq \bar{y}$, $Var(L) > Var(\bar{y})$ ∎

(2) (a) $\ell' b$

(b) $E(\ell' b) = \ell' E(b) = \ell' \beta$

(c) $\ell' b = \underbrace{\ell' (X'X)^{-1} X'}_{c_0'} y$, so

$c_0 = X(X'X)^{-1}\ell$

(d)(i) $Var(c' y) = cov(c' y) = c' cov(y) c$
$= c' \sigma^2 I_n c = \sigma^2 c'c$ same as 1d

(ii) $Var(c_0' y) = cov(c_0' y) = cov(\ell' b)$
$= \ell' \sigma^2 (X'X)^{-1} \ell = \sigma^2 \ell' (X'X)^{-1} \ell$

(e) $E(c' y) = c' E(y) = c' X\beta = \ell' \beta$

$c'X$ and ℓ' are both $1 \times (k+1)$ vectors. Let I_j denote a $(k+1)\times 1$ vector with a one in position j and the rest zeros. If $c'X\beta = \ell'\beta$ for all $\beta \in \mathbb{R}^{k+1}$, this includes I_j, and $c'X I_j = \ell' I_j$ for $j = 1, \ldots, k+1$. Thus the first element of $c'X$ is ℓ_1, the second is ℓ_2, ... and $\boxed{c'X = \ell' \Leftrightarrow \ell = X'c}$

(2f) From (e) $X'c = \ell$

$$\Rightarrow \underbrace{X(X'X)^{-1}X'c}_{= Hc} = \underbrace{X(X'X)^{-1}\ell}_{= c_0} \leftarrow 2c$$

(g) $\mathrm{Var}(c'y) - \mathrm{Var}(c_0'y) = \mathrm{cov}(c'y) - \mathrm{cov}(c_0'y)$

$$= c'\sigma^2 I_n c - c_0'\sigma^2 I_n c_0$$

$$\overset{f}{=} \sigma^2\left(c'Ic - (Hc)'Hc\right)$$

$$= \sigma^2\left(c'Ic - c'H'Hc\right)$$

$$= \sigma^2 c'(I-H)c = \sigma^2 c'(I-H)'\underbrace{(I-H)c}_{n \times 1}$$

$$= \sigma^2 z'z \geq 0$$

(h) If $\mathrm{Var}(c'y) = \mathrm{Var}(c_0'y)$ then

$$z = (I-H)c = 0 = c - Hc = c - c_0$$

$$\Rightarrow c = c_0$$

(3) (a) $Var(b) = Var\left(\frac{\sum x_i y_i}{\sum x_i^2} \right)$

$= \frac{1}{(\sum x_i^2)^2} \sum_{i=1}^{n} x_i^2 \, Var(y_i) = \frac{\sigma^2 \sum_{i=1}^{n} x_i^2}{(\sum x_i^2)^2}$

$= \frac{\sigma^2}{\sum x_i^2}$

(b) (i) $E(b_2) = E\left(\frac{\bar{y}}{\bar{x}}\right) = \frac{1}{\bar{x}} E(\bar{y})$

$= \frac{1}{\bar{x}} \frac{1}{n} \sum_{i=1}^{n} E(y_i) = \frac{1}{n\bar{x}} \sum_{i=1}^{n} x_i \beta = \frac{\bar{x}}{\bar{x}} \beta$

$= \beta$ unbiased

(ii) Yes. $\frac{\bar{y}}{\bar{x}} = \frac{\sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i}$, so

$c_i = \frac{1}{\sum_{j=1}^{n} x_j}$

(iii) $Var(b_2) = \frac{1}{\bar{x}^2} \frac{1}{n^2} \sum_{i=1}^{n} Var(y_i)$

$= \frac{\cancel{n} \sigma^2}{\cancel{n} n \bar{x}^2} = \frac{\sigma^2}{n \bar{x}^2}$

(iv) b_2 is a linear unbiased estimator, and Gauss-Markov says b is BLUE.

(3c) $b_3 = \frac{1}{n}\sum_{i=1}^{n} \frac{y_i}{x_i}$

(i) $E(b_3) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{x_i} E(y_i) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\cancel{x_i}} \beta \cancel{x_i} = \frac{n\beta}{n} = \beta$

Yes

(ii) Yes, with $c_i = \frac{1}{n x_i}$

(iii) $Var(b_3) = \frac{1}{n^2}\sum_{i=1}^{n} \frac{1}{x_i^2} Var(y_i) = \frac{\sigma^2}{n^2}\sum_{i=1}^{n} \frac{1}{x_i^2}$

(iv) Again, by Gauss-Markov.

(4) Residuals all zero mean that the points in the scatterplot all lie on a straight line — or in higher dimension, all on a single hyper-plane. If $y_1 \dots y_n$ have a joint density, this is an event of probability zero. Even if the distribution of $\underset{\sim}{y}$ is discrete, it is an event of very low probability.

(5) $He = X(X'X)^{-1}\underbrace{X'e}_{0} = 0_{n\times 1}$, so if H^{-1} exists

$H^{-1}He = H^{-1}0 = 0 \Rightarrow e = 0$

(6) $H = X(X'X)^{-1}X'$ The rank of X, X' and $(X'X)^{-1}$ are all $k+1$, so the rank of the $n\times n$ matrix H is $k+1$. If $k+1 < n$, cols of H are linearly dependent, and there is $v \neq 0$ with $Hv = 0$. If H had an inverse, $H^{-1}Hv = H^{-1}0 \Rightarrow v = 0$ but $v \neq 0$. This contradiction shows H^{-1} cannot exist.

(7) False. All you need to show this is one example. See p. 3 of the "Least Squares with R" lecture.

(8) True. $E(e) = E(y - \hat{y}) = X\beta - X\beta = 0_{n \times 1}$ so the sum of expected residuals is zero too.

(9) True

(10)
(a) $b_2 = (X'X)^{-1}X'\hat{y} = (X'X)^{-1}X'Xb = b$

(b) $\hat{\hat{y}} = Xb_2 = Xb = \hat{y}$

It's not surprising because $\hat{y}$, being the projection of y onto the space spanned by the columns of X, is in that space. So projecting it onto V again just yields the point $\hat{y}$ again.

(11) $b_3 = (X'X)^{-1}\underbrace{X'e}_{0} = 0_{(k+1)\times 1}$

$\hat{\hat{y}} = Xb_3 = X0 = 0_{n \times 1}$

(12) (a) $M_{Ax}(t) = E(e^{t'Ax}) = E(e^{(A't)'x})$

$= M_x(A't)$

(b) $M_{x+c}(t) = E(e^{t'(x+c)})$

$= E(e^{t'x + t'c}) = E(e^{t'x} e^{t'c})$

$= e^{t'c} E(e^{t'x}) = e^{t'c} M_x(t)$

(13) (a) $M_{\underset{\sim}{z}}(\underset{\sim}{t}) \overset{\text{ind}}{=} \prod_{i=1}^{p} M_{z_i}(t_i) = \prod_{i=1}^{p} e^{0 t_i + \frac{1}{2}\cdot 1 \cdot t_i^2}$

$= \prod_{i=1}^{p} e^{\frac{1}{2} t_i^2} = e^{\frac{1}{2}\sum_{i=1}^{p} t_i^2} = e^{\frac{1}{2} t' t}$

(b) $y = \Sigma^{1/2} z + \mu$

(i) $E(y) = \Sigma^{1/2} E(z) + \mu = 0 + \mu = \mu$

(ii) $\mathrm{cov}(y) = \mathrm{Cov}(\Sigma^{1/2} z) = \Sigma^{1/2} \mathrm{cov}(z) \Sigma^{1/2'}$

$= \Sigma^{1/2} I (\Sigma')^{1/2} = \Sigma^{1/2} \Sigma^{1/2} = \Sigma$

(iii) $M_y(t) = M_{\Sigma^{1/2} z + \mu}(t) \overset{12b}{=} e^{t'\mu} M_{\Sigma^{1/2} z}(t)$

$\overset{12a}{=} e^{t'\mu} M_z(\Sigma^{1/2'} t) = e^{t'\mu} e^{\frac{1}{2}(\Sigma^{1/2'} t)'(\Sigma^{1/2'} t)}$

$= e^{t'\mu + \frac{1}{2} t' \Sigma^{1/2} \Sigma^{1/2'} t} = e^{t'\mu + \frac{1}{2} t' \Sigma t}$

(14) $y \sim N_p(\mu, \Sigma)$

(a) $M_{Ay}(t) = M_y(A't)$

$= e^{(A't)'\mu + \frac{1}{2}(A't)'\Sigma(A't)}$

$= e^{t'(A\mu) + \frac{1}{2}t'(A\Sigma A')t}$

MGF of $N_r(A\mu, A\Sigma A')$

(b) $M_{y+c}(t) = e^{t'c} M_y(t)$

$= e^{t'c} e^{t'\mu + \frac{1}{2}t'\Sigma t}$

$= e^{t'(\mu+c) + \frac{1}{2}t'\Sigma t}$

MGF of $N_p(\mu+c, \Sigma)$

(15) Marginals are normal, with

$$M_{y_1}(t_1) = e^{\mu_1 t_1 + \frac{1}{2}\sigma_1^2 t_1^2} \quad \text{and}$$

$$M_{y_2}(t) = e^{\mu_2 t_2 + \frac{1}{2}\sigma_2^2 t_2^2}$$

$$M_y(t) = e^{t'\mu + \frac{1}{2}t'\Sigma t}$$

$$= \operatorname{Exp}\left\{ t_1\mu_1 + t_2\mu_2 + \frac{1}{2}(t_1, t_2)\begin{pmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{pmatrix}\begin{pmatrix}t_1\\ t_2\end{pmatrix}\right\}$$

$$= e^{\mu_1 t_1} e^{\mu_2 t_2} \operatorname{Exp}\left\{\frac{1}{2}(t_1\sigma_1^2, t_2\sigma_2^2)\begin{pmatrix}t_1\\ t_2\end{pmatrix}\right\}$$

$$= e^{\mu_1 t_1} e^{\mu_2 t_2} \operatorname{Exp}\left\{\frac{1}{2}(t_1\sigma_1^2 t_1 + t_2\sigma_2^2 t_2)\right\}$$

$$= e^{\mu_1 t_1} e^{\mu_2 t_2} e^{\frac{1}{2}\sigma_1^2 t_1^2} e^{\frac{1}{2}\sigma_2^2 t_2^2}$$

$$= e^{\mu_1 t_1 + \frac{1}{2}\sigma_1^2 t_1^2} \; e^{\mu_2 t_2 + \frac{1}{2}\sigma_2^2 t_2^2}$$

$$= M_{y_1}(t_1) \qquad M_{y_2}(t_2)$$

$\Longrightarrow$ independent. $\boxtimes$

(16)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$y = A x$$

$$y \sim N(A\mu, A\Sigma A'), \quad A\mu = \begin{pmatrix} \mu_1 + \mu_2 \\ \mu_2 + \mu_3 \end{pmatrix}$$

$$A\Sigma A' =$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix},$$

So

$$y \sim N_2\left(\begin{pmatrix} 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 4 \end{pmatrix} \right)$$

(17)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$y = A \quad x$, so $y \sim N(A\mu, A\Sigma A')$

$$A\mu = \begin{pmatrix} \mu_1 + \mu_2 \\ \mu_1 - \mu_2 \end{pmatrix}$$

$$A\Sigma A' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & \sigma_2^2 \\ \sigma_1^2 & -\sigma_2^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 + \sigma_2^2 & \sigma_1^2 - \sigma_2^2 \\ \sigma_1^2 - \sigma_2^2 & \sigma_1^2 + \sigma_2^2 \end{pmatrix}$$

For independence, need $\sigma_1^2 = \sigma_2^2$

(18) (a) $y \sim N(a\mu + b, a^2\sigma^2)$

(b) $z \sim N(0,1)$

(c) $y \sim N(n\mu, n\sigma^2)$

(d) $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$

(e) $z \sim N(0,1)$

(f) $y \sim N(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2)$

(g) $y \sim \chi^2(\sum_{i=1}^{n} r_i)$

(h) $z \sim \chi^2(1)$

(i) $y \sim \chi^2(n)$

(j) $x_2 \sim \chi^2(r_2)$

(19) Is computer - no solution given.