Assignment 3

(1) a (2) c (3) b (4) b

(5) c (6) b (7) d (8) d

(9) $tr(AB) = tr\left(\sum_{k=1}^{q} a_{ik} b_{kj}\right) = \sum_{i=1}^{p} \sum_{k=1}^{q} a_{ik} b_{ki}$

switch both

$= \sum_{k=1}^{q} \sum_{i=1}^{p} b_{ki} a_{ik} = tr(BA)$ ∎

There was more detail in the lecture slides. This is enough.

(10) If $AB = I$, both A and B must have inverses, for otherwise

$$|AB| = |A||B| = 0 \neq 1 = |I|$$

Then $AB = I \Rightarrow A^{-1}AB = A^{-1}I \Rightarrow B = A^{-1}$ and
$AB = I \Rightarrow ABB^{-1} = IB^{-1} \Rightarrow A = B^{-1}$ ∎

(11) $BA = I \Rightarrow \underbrace{BA}_{I}C = IC \Rightarrow B = C$ ∎

(12) $B^{-1}A^{-1}(AB) = B^{-1}IB = I$ ∎

(13) Denoting $a = [a_j]$, $a'a = \sum_{j=1}^{n} a_j^2 \geq 0$

(14) Suppose on the contrary that A^{-1} does exist. Columns of X linearly dependent means there exists $v \neq 0$ with $Av = 0 \Rightarrow A^{-1}Av = A^{-1}0$ $\Rightarrow v = 0$. But this is impossible since $v \neq 0$.

∎

(15) Let $B = A^{-1}$. Seek to show B' is the inverse of A': $B'A' = (AB)' = I' = I$ ☐

(16) If A is symmetric $(A^{-1})' = (A')^{-1} = A^{-1}$ ☐

(17) $\Sigma v = \lambda v \implies v'\Sigma v = v'\lambda v = \lambda v'v = \lambda$

Since $v'\Sigma v > 0$ and $v'\Sigma v = \lambda$, have $\lambda > 0$ ☐

(18) (a) (i) D^{-1} is a diagonal matrix with jth diagonal element $1/\lambda_i$

(ii) $C D^{-1} \underbrace{C' C}_{I} D C' = C \underbrace{D^{-1} D}_{I} C' = CC' = I$

(b) (i) $D^{1/2}$ is a diagonal matrix with jth diagonal element $\sqrt{\lambda_i}$.

(ii) $(C D^{1/2} C')' = C'' D^{1/2\,\prime} C' = C D^{1/2} C'$

(iii) $\Sigma^{1/2} \Sigma^{1/2} = C D^{1/2} \underbrace{C'C}_{I} D^{1/2} C'$

$= C D^{1/2} D^{1/2} C' = C D C' = \Sigma$

(c) (i) $\Sigma^{-1/2} \Sigma^{1/2} = \underbrace{C \underbrace{D^{-1/2} \underbrace{C' C}_{I} D^{1/2}}_{I} C}_{I} = I$ ☐

(ii) $\Sigma^{-1/2} \Sigma^{-1/2} = C D^{-1/2} \underbrace{C' C}_{I} D^{-1/2} C'$

$= C D^{-1/2} D^{-1/2} C' = C D^{-1} C'$

$= \Sigma^{-1}$ by part (a)

(18b) The eigenvalues of Σ are strictly positive by problem 17, and 18 a (ii) gives a formula for the inverse.

(19) Let v be all zeros except for a 1 in position j.
Then $v' A v = a_{jj} > 0$ since A is positive definite.

(20) $\operatorname{tr}(M) = \operatorname{tr}(\underset{A}{C}\underbrace{DC'}_{B}) = \operatorname{tr}(D\underbrace{C'C}_{I}) = \operatorname{tr}(D)$

$= \sum_{j=1}^{p} \lambda_j$

(21) $|M| = |CDC'| = |DC'C| = |D|$

$= \prod_{j=1}^{p} \lambda_j$

(22) $E(Ax) = AE(x) = A\mu$

$\mathrm{Cov}(Ax) = E\{(Ax - A\mu)(Ax - A\mu)'\}$

$= E\{A(x-\mu)(A(x-\mu))'\}$

$= E\{A(x-\mu)(x-\mu)'A'\} = AE\{(x-\mu)(x-\mu)'\}A'$

$= A\Sigma A'$

(23) $\mathrm{Cov}(Ay, By) = E\{(Ay - A\mu)(By - B\mu)'\}$

$= E\{A(y-\mu)(B(y-\mu))'\} = E\{A(y-\mu)(y-\mu)'B'\}$

$= A\,E\{(y-\mu)(y-\mu)'\}B' = A\Sigma B'$

(24) $\Sigma = E\{(x-\mu)(x-\mu)'\}$

$= E\{xx' - x\cancel{\mu'} - \cancel{\mu}x' + \cancel{\mu\mu'}\}$

$= E\{xx'\} - E\{x\}\cancel{\mu'} - \cancel{\mu}E\{x\} + \cancel{\mu\mu'}$

$= E\{xx'\} - \cancel{\mu\mu'} - \cancel{\mu\mu'} + \cancel{\mu\mu'}$

$= E\{xx'\} - \mu\mu'$

(25) Just kidding, but some students do this.

(26) $E(Ax+c) = A\mu + c$, and

$\mathrm{Cov}(Ax+c) = E\{(Ax+c-(A\mu+c))(Ax+c-(A\mu+c))'\}$

$= E\{(Ax - A\mu)(Ax - A\mu)'\}$ Already did this

$= A\Sigma A'$

(27) Because variances cannot be negative

$$0 \le Var(y) = cov(v'x) = v' cov(x) v$$
$$= v' \Sigma v \quad \text{where} \quad \Sigma = cov(x)$$

Because v could be anything in $\mathbb{R}^p$, that's it.

(28) (a) Let $cov(y) = \Sigma$. If λ is an eigenvalue of Σ and v the corresponding eigenvector,

$$\Sigma v = \lambda v \Rightarrow v'\Sigma v = v'\lambda v = \lambda v'v = \lambda \ge 0$$

by problem 27.

(b) $|\Sigma| \ge 0$ because the determinant is the product of the eigenvalues.

(c) Denote $cov\begin{pmatrix} x \\ y \end{pmatrix}$ by $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$. By (b),

$$0 \le |\Sigma| = \sigma_1^2 \sigma^2 - \sigma_{12}^2$$

$$\Rightarrow \underbrace{\sigma_{12}^2 \le \sigma_1^2 \sigma_2^2}_{\text{Cauchy-Schwarz}} \Rightarrow \frac{\sigma_{12}^2}{\sigma_1^2 \sigma_2^2} \le 1$$

$$\Rightarrow \sqrt{\frac{\sigma_{12}^2}{\sigma_1^2 \sigma_2^2}} \le 1 \Rightarrow \Bigg| \underbrace{\frac{\sigma_{12}}{\sigma_1 \sigma_2}}_{corr(x,y)} \Bigg| \le 1$$

$$\Rightarrow -1 \le Corr(x, y) \le 1$$

29 (a) $\mathrm{Cov}(x_i, y_j)$

(b) $\mathrm{cov}(x+y) = E\{(x+y-(\mu_x+\mu_y))(x+y-(\mu_x+\mu_y))'\}$

$= E\{(x-\mu_x + y-\mu_y)(x-\mu_x + y-\mu_y)'\}$

$= E\{(x-\mu_x)(x-\mu_x)' + (x-\mu_x)(y-\mu_y)' + (y-\mu_y)(x-\mu_x)'$
$+ (y-\mu_y)(y-\mu_y)'\}$

$= E\{(x-\mu_x)(x-\mu_x)'\} + E\{(x-\mu_x)(y-\mu_y)'\}$
$+ E\{(y-\mu_y)(x-\mu_x)'\} + E\{(y-\mu_y)(y-\mu_y)'\}$

$= \mathrm{cov}(x) + \mathrm{cov}(x, y) + \mathrm{cov}(y, x) + \mathrm{cov}(y)$

(c) They are all $p \times p$.

(d) If $\mathrm{cov}(x, y) = 0$, $\mathrm{cov}(x+y) = \mathrm{cov}(x) + \mathrm{cov}(y)$

(e) $\mathrm{Cov}(x+c, y+d) = E\{(x+c-(\mu_x+c))(y+d-(\mu_y+d))'\}$

$= E\{(x+c-\mu_x-c)(y+d-\mu_y-d)'\}$

$= E\{(x-\mu_x)(y-\mu_y)'\} = \mathrm{cov}(x, y)$

(f) It's not true even if $p=q$. There is no reason why $\mathrm{cov}(x_i, y_j)$ should equal $\mathrm{cov}(x_j, y_i)$.

(g) This was supposed to be a joke.