STA302 F17 Assignment 2

(1) (a) $1-\alpha = P\left\{-t_{\alpha/2} < \frac{\sqrt{n}(\bar{y}-\mu)}{s} < t_{\alpha/2}\right\}$

$= P\left\{-t_{\alpha/2}\frac{s}{\sqrt{n}} < \bar{y}-\mu < t_{\alpha/2}\frac{s}{\sqrt{n}}\right\}$

$= P\left\{-\bar{y}-t_{\alpha/2}\frac{s}{\sqrt{n}} < -\mu < -\bar{y}+t_{\alpha/2}\frac{s}{\sqrt{n}}\right\}$

$= P\left\{\bar{y}+t_{\alpha/2}\frac{s}{\sqrt{n}} > \mu > \bar{y}-t_{\alpha/2}\frac{s}{\sqrt{n}}\right\}$

$= P\left\{\bar{y}-t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{y}+t_{\alpha/2}\frac{s}{\sqrt{n}}\right\}$

(b) $\bar{y}-t_{\alpha/2}\frac{s}{\sqrt{n}} = 2.57-(2.07)\sqrt{\frac{5.85}{23}}$

$= 2.57 - 1.04 = 1.53$

$\bar{y}+t_{\alpha/2}\frac{s}{\sqrt{n}} = 2.57+1.04 = 4.61$

So the 95% CI is $(1.53, 4.61)$

(c) (i) $T = \frac{\bar{y}-\mu_0}{s/\sqrt{n}} = \frac{2.57-3}{0.5043} = \frac{-0.43}{0.5043}$

$= -0.853$

(ii) No, don't reject H_0

(iii) No

(iv) NA

(2) (a) $E(y_i) = E(\beta_0 + \beta_1 x_i + \varepsilon) = E(\beta_0 + \beta_1 x_i) + E(\varepsilon_i)$

$$= \beta_0 + \beta_1 x_i + 0 = \beta_0 + \beta_1 x_i$$

(b) $\frac{\partial S}{\partial \beta_0} = \frac{\partial}{\partial \beta_0} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$

$$= \sum_{i=1}^{n} 2(y_i - \beta_0 - \beta_1 x_i)(-1) \overset{set}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} y_i - n\beta_0 - \beta_1 \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow n\beta_0 = \sum_{i=1}^{n} y_i - \beta_1 \sum_{i=1}^{n} x_i \Rightarrow \beta_0 = \bar{y} - \beta_1 \bar{x} \quad (*)$$

$$\frac{\partial S}{\partial \beta_1} = \frac{\partial}{\partial \beta_1} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

$$= \sum_{i=1}^{n} 2(y_i - \beta_0 - \beta_1 x_i)(-x_i)$$

$$= -2 \sum_{i=1}^{n} (x_i y_i - x_i \beta_0 - \beta_1 x_i^2)$$

$$= -2 \left(\sum_{i=1}^{n} x_i y_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 \right) \overset{set}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0$$

Substituting for β_0 using the first equation,

$$\sum_{i=1}^{n} x_i y_i - (\bar{y} - \beta_1 \bar{x}) n\bar{x} - \beta_1 \sum_{i=1}^{n} x_i^2 = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} + \beta_1 n\bar{x}^2 - \beta_1 \sum_{i=1}^{n} x_i^2 = 0$$

(2b continued)

$$\Rightarrow \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} - \beta_1 \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \right) = 0$$

$$\Rightarrow \beta_1 \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \right) = \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}$$

$$\Rightarrow \beta_1 = \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

Denoting this solution by b_1 gives (1.13)

Substituting b_1 for β_1 in (*) gives

$$b_0 = \bar{y} - b_1 \bar{x}, \text{ which is } (1.11)$$

2c

(1.14) says $b_1 = \dfrac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$

$$= \frac{\sum_{i=1}^{n}(x_i y_i - x_i\bar{y} - \bar{x}y_i + \bar{x}\bar{y})}{\sum_{i=1}^{n}(x_i^2 - 2x_i\bar{x} + \bar{x}^2)}$$

$$= \frac{\sum_{i=1}^{n}x_i y_i - \bar{y}\sum_{i=1}^{n}x_i - \bar{x}\sum_{i=1}^{n}y_i + n\bar{x}\bar{y})}{\sum_{i=1}^{n}x_i^2 - 2\bar{x}\sum_{i=1}^{n}x_i + n\bar{x}^2}$$

$$= \frac{\sum_{i=1}^{n}x_i y_i - n\bar{x}\bar{y} - n\bar{x}\bar{y} + n\bar{x}\bar{y}}{\sum_{i=1}^{n}x_i^2 - 2n\bar{x}^2 + n\bar{x}^2}$$

$$= \frac{\sum_{i=1}^{n}x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n}x_i^2 - n\bar{x}^2}$$ which is (1.13)

(d) $\sum_{i=1}^{n}(x_i-\bar{x}) = \sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\bar{x} = \sum_{i=1}^{n}x_i - n\bar{x}$

$= \sum_{i=1}^{n}x_i - \cancel{n}\dfrac{\sum_{i=1}^{n}x_i}{\cancel{n}} = 0$

(e) $\dfrac{\partial}{\partial \gamma_0}\sum_{i=1}^{n}\left(y_i-\gamma_0-\beta_1(x_i-\bar{x})\right)^2$

$= 2\sum_{i=1}^{n}(y_i-\gamma_0-\beta_1(x_i-\bar{x}))(-1) \overset{\text{set}}{=} 0$

$\Longrightarrow \sum_{i=1}^{n}y_i - n\gamma_0 - \beta_1\underbrace{\sum_{i=1}^{n}(x_i-\bar{x})}_{=0 \text{ by (d)}} = 0$

$\Longrightarrow \sum_{i=1}^{n}y_i = n\gamma_0 \Longrightarrow \gamma_0 = \dfrac{\sum_{i=1}^{n}y_i}{n} = \bar{y}$

And the least squares estimate of γ_0 is $\bar{y}$ for <u>any</u> β_1

(2e continued)

$$\frac{\partial}{\partial \beta_1} \sum_{i=1}^{n} \left(y_i - \gamma_0 - \beta_1 (x_i - \bar{x}) \right)^2$$

$$= 2 \sum_{i=1}^{n} \left(y_i - \gamma_0 - \beta_1 (x_i - \bar{x}) \right) (-1) (x_i - \bar{x}) \overset{set}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} \left(y_i - \gamma_0 - \beta_1 (x_i - \bar{x}) \right) (x_i - \bar{x}) = 0$$

$$\Rightarrow \sum_{i=1}^{n} (y_i - \gamma_0)(x_i - \bar{x}) - \beta_1 \sum_{i=1}^{n} (x_i - \bar{x})^2 = 0$$

substituting $\gamma_0 = \bar{y}$ from the first equation,

$$\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \beta_1 \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$\Rightarrow \beta_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \quad \text{which is (1.14)}$$

And the least squares estimate of the slope b_1 is unaffected by centering

(f) $$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - \hat{y}_i) = \sum_{i=1}^{n} \left(y_i - (b_0 + b_1 x_i) \right)$$

$$= \sum_{i=1}^{n} \left(y_i - (\bar{y} - b_1 \bar{x}) - b_1 x_i \right)$$

$$= \sum_{i=1}^{n} \left(y_i - \bar{y} - b_1 (x_i - \bar{x}) \right) =$$

$$= \underbrace{\sum_{i=1}^{n} (y_i - \bar{y})}_{0} - b_1 \underbrace{\sum_{i=1}^{n} (x_i - \bar{x})}_{0} = 0$$

(2g) Exercise 1.6, p. 23

$y_i = \mu + \varepsilon_i$, $E(\varepsilon_i) = 0$, $Var(\varepsilon_i) = \sigma^2$, y_i s independent

$E(y_i) = \mu$, minimize $S = \sum_{i=1}^{n} (y_i - \mu)^2$

$$\frac{dS}{d\mu} = \frac{d}{d\mu} \sum_{i=1}^{n} (y_i - \mu)^2 = \sum_{i=1}^{n} 2(y_i - \mu)(-1) \overset{\text{set}}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} (y_i - \mu) = 0 \Rightarrow \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} \mu = \sum_{i=1}^{n} y_i - n\mu = 0$$

$$\Rightarrow \sum_{i=1}^{n} y_i = n\mu \Rightarrow \mu = \bar{y}$$

$$Var(\bar{y}) = Var\left(\frac{1}{n} \sum_{i=1}^{n} y_i\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} y_i\right)$$

$$\overset{\text{ind}}{=} \frac{1}{n^2} \sum_{i=1}^{n} Var(y_i) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 = \frac{1}{n^2} n\sigma^2$$

$$= \frac{\sigma^2}{n}$$

You don't have to show quite this much work.

(h) $b_0 = \cancel{-1.6384}$ $\boxed{-1.60384}$ $b_1 = \cancel{0.3264}$ $\boxed{0.03264}$

(3) (a) $E(y_i) = E(\beta_1 x_i + \varepsilon_i) = E(\beta_1 x_i) + E(\varepsilon_i)$

$$= \beta_1 x_i + 0 = \beta_1 x_i$$

(b) $$\frac{d}{d\beta_1} \sum_{i=1}^{n} (y_i - \beta_1 x_i)^2 = 2 \sum_{i=1}^{n} (y_i - \beta_1 x_i)(-x_i)$$

$$= -2 \sum_{i=1}^{n} (x_i y_i - \beta_1 x_i^2) = -2\left(\sum_{i=1}^{n} x_i y_i - \beta_1 \sum_{i=1}^{n} x_i^2\right)$$

$$\overset{\text{set}}{=} 0 \Rightarrow \sum_{i=1}^{n} x_i y_i = \beta_1 \sum_{i=1}^{n} x_i^2$$

$$\Rightarrow \beta_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$ call this solution b_1

2nd Derivative test

$$\frac{d}{d\beta_1} -2\left(\sum_{i=1}^{n} x_i y_i - \beta_1 \sum_{i=1}^{n} x_i^2\right)$$

$$= (-2)\left(-\sum_{i=1}^{n} x_i^2\right) = 2 \sum_{i=1}^{n} x_i^2 > 0$$

Concave up, minimum

(c) (Exercise 1.1) $$E(b_1) = E\left\{\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}\right\}$$

$$= \frac{\sum_{i=1}^{n} x_i E(y_i)}{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n} x_i (x_i \beta_1)}{\sum_{i=1}^{n} x_i^2} = \beta_1 \frac{\sum x_i^2}{\sum x_i^2} = \beta_1$$

(3d) $\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - \hat{y}_i) = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} b_1 x_i$

$$= \sum_{i=1}^{n} y_i - \frac{\left(\sum_{i=1}^{n} x_i y_i\right)\left(\sum_{i=1}^{n} x_i\right)}{\sum_{i=1}^{n} x_i^2}$$

No, no particular reason it should be zero

(e) $E(e_i) = E(y_i - \hat{y}_i) = E(y_i) - E(b_1 x_i)$

$$= x_i \beta - x_i E(b_1) = x_i \beta - x_i \beta = 0$$

$E(\sum_{i=1}^{n} e_i) = \sum_{i=1}^{n} E(e_i) = \sum_{i=1}^{n} 0 = 0$

That's the easy way, but also taking the expected value of the messy expression in (3d) yields zero.

(4) (a) $M_{ax}(t) = E(e^{(ax)t}) = E(e^{x(at)})$

$= M_x(at)$

(b) $M_{x+a}(t) = E(e^{(x+a)t}) = E(e^{xt+at})$

$= E(e^{xt}e^{at}) = e^{at}E(e^{xt})$

$= e^{at}M_x(t)$

(c) $M_{\sum_{i=1}^n x_i}(t) = E(e^{(\sum_{i=1}^n x_i)t}) = E(e^{\sum_{i=1}^n (x_i t)})$

$= E\left(\prod_{i=1}^n e^{x_i t}\right)$

$= \int \cdots \int e^{x_1 t} e^{x_2 t} \cdots e^{x_n t} f_{\underset{\sim}{x}}(x_1, x_2 \cdots x_n)\, dx_1\, dx_2 \cdots dx_n$

$\overset{\text{ind}}{=} \int \cdots \int e^{x_1 t} \cdots e^{x_n t} f_{x_1}(x_1) \cdots f_{x_n}(x_n)\, dx_1 \cdots dx_n$

$= \int \cdots \int e^{x_2 t} \underbrace{\left(\int e^{x_1 t} f_{x_1}(x_1)\, dx_1\right)}_{M_{x_1}(t)} f_{x_2}(x_2)\, dx_2 \cdots f_{x_n}(x_n)\, dx_n$

$= M_{x_1}(t) \int \cdots \int e^{x_3 t} \underbrace{\left(\int e^{x_2 t} f_{x_2}(x_2)\, dx_2\right)}_{M_{x_2}(t)} f_{x_3}(x_3)\, dx_3 \cdots f_{x_n}(x_n)\, dx_n$

4c continued

$= M_{x_1}(t)\, M_{x_2}(t) \cdots M_{x_n}(t)$ as claimed.

5 (a) $M_y(t) = M_{ax+b}(t) \overset{4a\,\&\,b}{=} e^{bt} M_x(at)$

$= e^{bt} e^{\mu(at) + \frac{1}{2}\sigma^2(at)^2}$

$= e^{(a\mu+b)t + \frac{1}{2}(a^2\sigma^2)t^2}$

MGF of $N(a\mu+b, a^2\sigma^2)$

(b) $z = \frac{x-\mu}{\sigma}$. This is (5a) with $a = \frac{1}{\sigma}$, $b = -\frac{\mu}{\sigma}$

so $z \sim N\left(\frac{1}{\sigma}\mu - \frac{\mu}{\sigma}, \frac{1}{\sigma^2}\sigma^2\right) = N(0,1)$

(c) $M_y(t) = M_{\Sigma x_i}(t) \overset{4c}{=} \prod_{i=1}^{n} M_{x_i}(t)$

$= \prod_{i=1}^{n} e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$= e^{(n\mu)t + \frac{1}{2}(n\sigma^2)t^2}$

MGF of $N(n\mu, n\sigma^2)$

(d) $M_{\bar{x}}(t) = M_{\frac{1}{n}\Sigma x_i}(t) \overset{4a}{=} M_{\Sigma x_i}\left(\frac{t}{n}\right)$

$= M_y\left(\frac{t}{n}\right)$ where $y = \sum x_i$ as in 5c

$= e^{n\mu \cdot \frac{t}{n} + \frac{1}{2} n\sigma^2 \cdot \frac{t^2}{n^2}} = e^{\mu t + \frac{1}{2}\left(\frac{\sigma^2}{n}\right)t^2}$

MGF of $N(\mu, \sigma^2/n)$

(5e) By 5d, $\bar{x} \sim N(\mu, \frac{\sigma^2}{n})$, and then by 5b $z = \frac{\sqrt{n}(\bar{x}-\mu)}{\sigma} = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$

$$= \frac{\bar{x} - E(\bar{x})}{\sqrt{Var(\bar{x})}} \sim N(0,1)$$

(f) By (5a), $a_i x_i \sim N(a_i \mu_i, a_i^2 \sigma_i^2)$, and then by (4c),

$$M_{\Sigma a_i x_i}(t) = \prod_{i=1}^{n} M_{a_i x_i}(t) = \prod_{i=1}^{n} e^{a_i \mu_i t + \frac{1}{2}(a_i^2 \sigma_i^2) t^2}$$

$$= e^{(\sum_{i=1}^{n} a_i \mu_i) t + \frac{1}{2}(\sum_{i=1}^{n} a_i^2 \sigma_i^2) t^2}$$

MGF of $N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$

(6) Model (1.20) is $y_i = \beta_1 x_i + \varepsilon_i$

If $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$, $y_i \sim N(\beta_1 x_i, \sigma^2)$

$b_1 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$ is a linear combination of y_i values as in (5f), so it's normal. $E(b_1) = \beta_1$ by 3c, and

6 continued

$$Var(b_1) = Var\left(\frac{\sum_{i=1}^{n} x_i y_i}{\sum x_i^2}\right)$$

$$\overset{ind}{=} \frac{1}{(\sum x_i^2)^2} \sum_{i=1}^{n} x_i^2 \, Var(y_i)$$

$$= \frac{\sigma^2 \sum x_i^2}{(\sum x_i^2)^2} = \frac{\sigma^2}{\sum x_i^2}, \text{ so}$$

$$b_1 \sim N\left(\beta_1, \frac{\sigma^2}{\sum x_i^2}\right)$$

(7) (a) $M_Y(t) = M_{\sum X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t)$

$$= \prod_{i=1}^{n} (1-2t)^{-r_i/2} = (1-2t)^{-(\sum_{i=1}^{n} r_i)/2}$$

MGF of $\chi^2\left(\sum_{i=1}^{n} r_i\right)$

7b

$$M_{z^2}(t) = E\left(e^{z^2 t}\right)$$

$$= \int_{-\infty}^{\infty} e^{z^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(1-2t)z^2}\, dz$$

$$= (1-2t)^{-1/2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{(1-2t)^{-1/2}\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{z^2}{(1-2t)^{-1}}}}_{}\, dz$$

Density of a normal with expected value zero & variance $(1-2t)^{-1}$, provided $(1-2t)^{-1} = \frac{1}{1-2t} > 0 \iff 1-2t > 0 \iff 1 > 2t \iff t < \frac{1}{2}$. If so the integral converges and $= 1$

$$= (1-2t)^{-\frac{1}{2}}, \quad \text{MGF of } \chi^2(1)$$

(7c) $y = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2$

$= \sum_{i=1}^{n} z_i^2$, where z_i are independent because the x_i are independent,

$z_i \sim N(0,1)$ by (5b), $z_i^2 \sim \chi^2(1)$ by (7b), and the sum of independent chi-squares is chi-squared by (7a) with

$df = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} 1 = n$, so

$y \sim \chi^2(n)$

(d) $M_y(t) = M_{x_1}(t) \, M_{x_2}(t)$

$M_y(t) = (1-2t)^{-\frac{r_1+r_2}{2}}$, $\quad M_{x_2}(t) = (1-2t)^{-r_2/2}$

$\Rightarrow (1-2t)^{-r_1/2} \cancel{(1-2t)^{-r_2/2}} = M_{x_1}(t) \cancel{(1-2t)^{-\frac{r_2}{2}}}$

$\Rightarrow M_{x_1}(t) = (1-2t)^{-r_1/2}$, MGF of $\chi^2(r_1)$

7e Following the hint,

$$\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}(\underbrace{x_i-\bar{x}}_{a_i}+\underbrace{\bar{x}-\mu}_{b})^2$$

$$= \sum_{i=1}^{n}\left((x_i-\bar{x})^2 + 2(\bar{x}-\mu)(x_i-\bar{x}) + (\bar{x}-\mu)^2\right)$$

$$= \sum_{i=1}^{n}(x_i-\bar{x})^2 + 2(\bar{x}-\mu)\underbrace{\sum_{i=1}^{n}(x_i-\bar{x})}_{=0 \text{ by } 2d} + n(\bar{x}-\mu)^2$$

$$= \sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sigma^2} + \frac{n(\bar{x}-\mu)^2}{\sigma^2}$$

That is

$$\underbrace{\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}_{\chi^2(n) \text{ by } (7c)} = \frac{(n-1)s^2}{\sigma^2} + \left(\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma}\right)^2$$

Inside is $N(0,1)$ by 5e, Square is $\chi^2(1)$ by 7b

The two terms on the right are independent because they are functions of s^2 and $\bar{x}$, which are independent. Then by (7d),

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

$(\nu_1+\nu_2 = n,\ \nu_2 = 1)$