# Assignment 1

(1)

```
> x = -4:4; px = abs(x)/20; y = x^2-1
> cbind(x,x^2,px,y)
       x      px  y
 [1,] -4 16 0.20 15
 [2,] -3  9 0.15  8
 [3,] -2  4 0.10  3
 [4,] -1  1 0.05  0
 [5,]  0  0 0.00 -1
 [6,]  1  1 0.05  0
 [7,]  2  4 0.10  3
 [8,]  3  9 0.15  8
 [9,]  4 16 0.20 15
> Ex = sum(x*px); Ex # You know it's zero by symmetry
[1] 0
> Varx = sum(x^2*px); Varx
[1] 10
> # Use formula for E(g(x)) to get E(y) and Var(y)
> Ey = sum(y*px); Ey
[1] 9
> Vary = sum( (y-Ey)^2 * px ); Vary
[1] 30
```

(f) $P(y=0) = P(x=1) + P(x=-1) = 1/20 + 1/20 = 1/10$

$P(y=3) = P(x=2) + P(x=-2) = 2/20 + 2/20 = 2/10$

$P(y=8) = P(x=3) + P(x=-3) = 3/20 + 3/20 = 3/10$

$P(y=15) = P(x=4) + P(x=-4) = 4/20 + 4/20 = 4/10$

(g) $E(y) = \sum_y y\,p(y) = 0 \cdot \frac{1}{10} + 3 \cdot \frac{2}{10} + 8 \cdot \frac{3}{10} + 15 \cdot \frac{4}{10}$

$= (6 + 24 + 60)/10 = 90/10 = 9$

(h) $Var(y) = \sum_y (y - \mu_y)^2 p(y)$

$= (0-9)^2 \cdot \frac{1}{10} + (3-9)^2 \cdot \frac{2}{10} + (8-9)^2 \cdot \frac{3}{10} + (15-9)^2 \cdot \frac{4}{10}$

$= 81 \cdot \frac{1}{10} + 36 \cdot \frac{2}{10} + 1 \cdot \frac{3}{10} + 36 \cdot \frac{4}{10}$

$= (81 + 72 + 3 + 144)/10 = 300/10 = 30$

Note (g) & (h) agree with R calculations.

2 (a) $E(x) = \sum_x x\, p(x) = a \cdot 1 = a$

$Var(x) = \sum_x (x-\mu_x)^2 p(x) = (a-a)^2 \cdot 1 = 0$

(b) $E(y) = \int_{-\infty}^{\infty} a\, f(x)\, dx = a \int_{-\infty}^{\infty} f(x)\, dx = a \cdot 1 = a$

$$Var(y) = E(y - E(y))^2 = E\{(g(x)-a)^2\} = E\{h(x)\}$$

$$= \int_{-\infty}^{\infty} (a-a)^2 f(x)\, dx = \int_{-\infty}^{\infty} 0 \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} 0\, dx = 0$$

(3) (a and b): Write the marginal distributions on the margins

| | $x=1$ | $x=2$ | $x=3$ | |
|---|---|---|---|---|
| $y=1$ | 3/12 | 1/12 | 3/12 | 7/12 |
| $y=2$ | 1/12 | 3/12 | 1/12 | 5/12 |
| | 4/12 | 4/12 | 4/12 | |

(c) $E(x) = (1+2+3)\cdot\frac{4}{12} = \frac{24}{12} = 2$

(d)

| $x$ | $x_c = x-2$ | $x_c^2 = (x-2)^2$ | $P(x)$ |
|---|---|---|---|
| 1 | -1 | 1 | 1/3 |
| 2 | 0 | 0 | 1/3 |
| 3 | 1 | 1 | 1/3 |

(i)

| $x_c$ | -1 | 0 | 1 |
|---|---|---|---|
| $P(x_c)$ | 1/3 | 1/3 | 1/3 |

(ii) $E(x_c) = (-1)\cdot\frac{1}{3} + (0)\cdot\frac{1}{3} + 1\cdot\frac{1}{3} = 0$

(iii)

| $x_c^2$ | $P(x_c^2)$ |
|---|---|
| 0 | 1/3 |
| 1 | 2/3 |

(iv) $E(x_c^2) = 0\cdot\frac{1}{3} + 1\cdot\frac{2}{3} = \frac{2}{3}$

(3e) $Var(x) = E(x_c^2) = \frac{2}{3}$

(f) $E(y) = 1 \cdot \frac{7}{12} + 2 \cdot \frac{5}{12} = \frac{17}{12}$

(g) $Var(y) = E(y^2) - (E(y))^2$

$E(y^2) = 1^2 \cdot \frac{7}{12} + 2^2 \cdot \frac{5}{12} = 7/12 + \frac{20}{12} = \frac{27}{12}$

$= 9/4$, so

$Var(y) = \frac{27}{12} - \left(\frac{17}{12}\right)^2 = \frac{324}{144} - \frac{289}{144} = \frac{35}{144}$

(h)

| $x$ | $y$ | $x+y$ | $P(x,y)$ |
|---|---|---|---|
| 1 | 1 | 2 | 3/12 |
| 1 | 2 | 3 | 1/12 |
| 2 | 1 | 3 | 1/12 |
| 2 | 2 | 4 | 3/12 |
| 3 | 1 | 4 | 3/12 |
| 3 | 2 | 5 | 1/12 |

| $z_1$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $P(z_1)$ | 3/12 | 2/12 | 6/12 | 1/12 |

(i) $E(z_1) = 2 \cdot \frac{3}{12} + 3 \cdot \frac{2}{12} + 4 \cdot \frac{6}{12} + 5 \cdot \frac{1}{2}$

$= (6 + 6 + 24 + 5)/12 = 41/12$

(j) $E(x) + E(y) = 2 + \frac{17}{12} = \frac{24+17}{12} = 41/12$ Yes

(3k)

| $x$ | $y$ | $xy$ | $P(x,y)$ |
|---|---|---|---|
| 1 | 1 | 1 | 3/12 |
| 1 | 2 | 2 | 1/12 |
| 2 | 1 | 2 | 1/12 |
| 2 | 2 | 4 | 3/12 |
| 3 | 1 | 3 | 3/12 |
| 3 | 2 | 6 | 1/12 |

| $z_2$ | 1 | 2 | 3 | 4 | 6 |
|---|---|---|---|---|---|
| $P(z_2)$ | 3/12 | 2/12 | 3/12 | 3/12 | 1/12 |

(l) $E(z_2) = 1 \cdot \frac{3}{12} + 2 \cdot \frac{2}{12} + 3 \cdot \frac{3}{12} + 4 \cdot \frac{3}{12} + 6 \cdot \frac{1}{12}$

$= (3+4+9+12+6)/12 = 34/12 = 17/6$

(m) $E(x)E(y) = 2 \cdot \frac{17}{12} = 17/6$ yes

(n) $Cov(x,y) = E(xy) - E(x)E(y) = \frac{17}{6} - \frac{17}{6} = 0$

(o) $x$ and $y$ independent means $P(x,y) = P(x)P(y)$ for all $x$ & $y$. $P(x=1, y=1) = 3/12 = 36/144$,

but $P(x=1)\,P(y=1) = \frac{4}{12} \cdot \frac{7}{12} = 28/144$

So the answer is <u>NO</u>, not independent.

There can be zero covariance without independence.

(4) $E(x_1 x_2) = \int\int x_1 x_2 \, f_{x_1, x_2}(x_1, x_2) \, dx_1 \, dx_2$

$= \int\int x_1 x_2 \, f_{x_1}(x_1) \, f_{x_2}(x_2) \, dx_1 \, dx_2$

This is where I use independence

$= \int x_2 \left( \int x_1 \, f_{x_1}(x_1) \, dx_1 \right) f_{x_2}(x_2) \, dx_2$

$= \int x_2 \, E(x_1) \, f_{x_2}(x_2) \, dx_2$

$= E(x_1) \int x_2 \, f_{x_2}(x_2) \, dx_2 = E(x_1) E(x_2)$

If $x_1$ & $x_2$ are discrete,

$E(x_1 x_2) = \sum_{x_1} \sum_{x_2} x_1 x_2 \, P(x_1, x_2)$

$= \sum_{x_1} \sum_{x_2} x_1 x_2 \, P_1(x_1) \, P_2(x_2)$

indep.

etc.

(5) (a) $\mathrm{Var}(y) = E\{(y-\mu_y)^2\}$

$= E\{y^2 - 2y\mu_y + \mu_y^2\} = E(y^2) - 2\mu_y E(y) + E(\mu_y^2)$

$= E(y^2) - 2\mu_y^2 + \mu_y^2 = E(y^2) - \mu_y^2$

(b) $\mathrm{Cov}(x,y) = E\{(x-\mu_x)(y-\mu_y)\}$

$= E\{xy - x\mu_y - \mu_x y + \mu_x\mu_y\}$

$= E(xy) - \mu_y E(x) - \mu_x E(y) + E(\mu_x\mu_y)$

$= E(xy) - 2\mu_x\mu_y + \mu_x\mu_y$

$= E(xy) - \mu_x\mu_y$

If $x$ & $y$ are independent, then $E(xy) = E(x)E(y)$ by Problem 4, and

$$\mathrm{Cov}(x,y) \underset{(b)}{=} E(xy) - E(x)E(y) = 0$$

Note however by Problem 3 that zero covariance does not necessarily imply independence.

(6) (a) Noting $E(ax) = a\mu_x$,

$$Var(ax) = E\{(ax - a\mu_x)^2\}$$
$$= E\{a^2(x-\mu_x)^2\} = a^2 E\{(x-\mu_x)^2\}$$
$$= a^2 \, Var(x)$$

(b) Noting $E(x+a) = \mu_x + a$,

$$Var(x+a) = E\{(x+a-(\mu_x+a))^2\}$$
$$= E\{(x+a-\mu_x-a)^2\}$$
$$= E\{(x-\mu_x)^2\} = Var(x)$$

(7) Noting $E(x+y) = \mu_x + \mu_y$,

$$Var(x+y) = E\{(x+y-(\mu_x+\mu_y))^2\}$$
$$= E\{((x-\mu_x)+(y-\mu_y))^2\}$$
$$= E\{(x-\mu_x)^2 + 2(x-\mu_x)(y-\mu_y) + (y-\mu_y)^2\}$$
$$= E\{(x-\mu_x)^2\} + 2E\{(x-\mu_x)(y-\mu_y)\} + E\{(y-\mu_y)^2\}$$
$$= Var(x) + 2\,cov(x,y) + Var(y)$$

(8) Noting $E(x+a)=\mu_x+a$ and $E(y+b)=\mu_y+b$,

$$Cov(x+a, y+b) = E\{(x+a-(\mu_x+a))(y+b-(\mu_y+b))\}$$

$$= E\{(x+a-\mu_x-a)(y+b-\mu_y-b)\}$$

$$= E\{(x-\mu_x)(y-\mu_y)\} = Cov(x,y)$$

(9) (a) $$Cov(ax, by) = E\{(ax-a\mu_x)(by-b\mu_y)\}$$

$$= E\{a(x-\mu_x)\,b(y-\mu_y)\}$$

$$= ab\,E\{(x-\mu_x)(y-\mu_y)\} = ab\,cov(x,y)$$

(b) $$Corr(ax, by) = \frac{cov(ax, by)}{\sqrt{Var(ax)\,Var(by)}}$$

$$= \frac{ab\,Cov(x,y)}{\sqrt{a^2\,Var(x)\;b^2\,Var(y)}}$$

$$= \frac{a}{|a|}\,\frac{b}{|b|}\,\frac{cov(x,y)}{\sqrt{Var(x)\,Var(y)}}$$

$= sign(a)\,sign(b)\,Corr(x,y)$,
provided $a \neq 0$ and $b \neq 0$

(10) $E(x_1 + x_2) = \sum_{x_1} \sum_{x_2} (x_1 + x_2)\, p(x_1, x_2)$

$$= \sum_{x_1} \sum_{x_2} \left( x_1 p(x_1, x_2) + x_2 p(x_1, x_2) \right)$$

$$= \sum_{x_1} \sum_{x_2} \underbrace{x_1}_{g_1(x_1, x_2)} p(x_1, x_2) + \sum_{x_1} \sum_{x_2} \underbrace{x_2}_{g_2(x_1, x_2)} p(x_1, x_2)$$

$$= E(x_1) + E(x_2)$$

If $x_1$ & $x_2$ are continuous,

$$E(x_1 + x_2) = \iint (x_1 + x_2)\, f(x_1, x_2)\, dx_1\, dx_2$$

$$= \iint x_1 f(x_1, x_2)\, dx_1\, dx_2 + \iint x_2 f(x_1, x_2)\, dx_1\, dx_2$$

$$= E(x_1) + E(x_2)$$

(11) (a) $E\left(\sum_{i=1}^{n} y_i\right) = \sum_{i=1}^{n} E(y_i) = \sum_{i=1}^{n} \mu$

$= n\mu$

(b) $Var\left(\sum_{i=1}^{n} y_i\right) = \sum_{i=1}^{n} Var(y_i)$

Problems 7 and 5c

$= n\sigma^2$

(c) $Var(\bar{y}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)$

$= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} y_i\right) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$

(d) $E(\bar{y}) = E\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(y_i)$

$= \frac{1}{n} \cdot n\mu = \mu$ unbiased

(e) If $\sum_{i=1}^{n} a_i = 1$, $E(L) = E\left(\sum_{i=1}^{n} a_i y_i\right) = \sum_{i=1}^{n} a_i E(y_i)$

$= \sum_{i=1}^{n} a_i \mu = \mu \sum_{i=1}^{n} a_i = \mu \cdot 1 = \mu$ unbiased

(f) Yes, $\bar{y} = \sum_{i=1}^{n} \left(\frac{1}{n}\right) y_i = \sum_{i=1}^{n} a_i y_i$, with

$a_i = \frac{1}{n}$ for $i = 1, \ldots, n$.

(g) $Var(L) = Var\left(\sum_{i=1}^{n} a_i y_i\right) = \sum_{i=1}^{n} a_i^2 Var(y_i)$

$= \sum_{i=1}^{n} a_i^2 \sigma^2 = \sigma^2 \sum_{i=1}^{n} a_i^2$

(12)

| | | |
|---|---|---|
| (a) No | (b) No | (c) Yes |
| (d) Yes | (e) No | (f) No |
| (g) Yes | (h) Yes | (i) No |

(13)

$$A'B = \begin{pmatrix} 2 & 1 & 0 \\ 5 & -4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2+2+0 & 0+3+0 \\ 5-8-3 & 0-12+9 \end{pmatrix}$$

(14) $c'd = 2\cdot 1 + 1\cdot 2 + 0\cdot -1 = 4$

$$cd' = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 & 2\cdot 2 & 2\cdot -1 \\ 1\cdot 1 & 1\cdot 2 & 1\cdot -1 \\ 0\cdot 1 & 0\cdot 2 & 0\cdot -1 \end{pmatrix}$$

(15)

(a) $AB = \begin{pmatrix} 4 & 4 \\ 8 & 8 \end{pmatrix} = AC$

(b) Yes

(c) That was a joke. $A^{-1}$ does not exist because the columns of A are linearly dependent.

(16)

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}$$

but

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 2 & 1 \end{pmatrix}$$

(17) $A^{-1}$ does not exist.

(18) $(X'X)' = X'X'' = X'X$

(19) The inverse is only defined for square matrices.