STA 302f17 Assignment One[1]

This course involves a lot of scalar calculations of expected value, variance and covariance, and even more matrix calculations. The questions on this assignment are pure review, and too basic for our textbook. Questions like these will not appear directly on the final exam. The formulas below will be supplied as needed on Quiz One.

$E(x) = \sum_x x\, p_x(x)$	$E(x) = \int_{-\infty}^{\infty} x f_x(x)\, dx$
$E(g(x)) = \sum_x g(x)\, p_x(x)$	$E(g(\mathbf{x})) = \sum_{x_1} \cdots \sum_{x_p} g(x_1, \ldots, x_p)\, p_{\mathbf{x}}(x_1, \ldots, x_p)$
$E(g(x)) = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx$	$E(g(\mathbf{x})) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_p)\, f_{\mathbf{x}}(x_1, \ldots, x_p)\, dx_1 \ldots dx_p$
$E(\sum_{i=1}^n a_i x_i) = \sum_{i=1}^n a_i E(x_i)$	$Var(x) = E\left((x - \mu_x)^2 \right)$
$Cov(x,y) = E\left((x-\mu_x)(y-\mu_y) \right)$	$Corr(x,y) = \frac{Cov(x,y)}{\sqrt{Var(x)Var(y)}}$

1. The discrete random variable x has probability mass function $p(x) = |x|/20$ for $x = -4, \ldots, 4$ and zero otherwise. Let $y = x^2 - 1$.
   (a) What is $E(x)$? The answer is a number. Show some work.
   (b) Calculate the variance of x. The answer is a number. My answer is 10.
   (c) What is $P(y = 8)$? My answer is 0.30
   (d) What is $P(y = -1)$? My answer is zero.
   (e) What is $P(y = -4)$? My answer is zero.
   (f) What is the probability distribution of y? Give the y values with their probabilities.

y	0	3	8	15
p(y)	0.1	0.2	0.3	0.4

   (g) What is $E(y)$? The answer is a number. My answer is 9.
   (h) What is $Var(y)$? The answer is a number. My answer is 30.

2. This question clarifies the meaning of $E(a)$ and $Var(a)$ when a is a constant.
   (a) Let x be a discrete random variable with $P(x = a) = 1$ (later we will call this a *degenerate* random variable). Using the definitions above, calculate $E(x)$ and $Var(x)$. This is the real meaning of the concept.
   (b) Let a be a real constant and x be a continuous random variable with density $f(x)$. Let $y = g(x) = a$. Using the formula for $E(g(x))$ above, calculate $E(y)$ and $Var(y)$. This reminds us that the change of variables formula (which is a very big theorem) applies to the case of a constant function.

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3. The discrete random variables x and y have joint distribution

	$x=1$	$x=2$	$x=3$
$y=1$	3/12	1/12	3/12
$y=2$	1/12	3/12	1/12

(a) What is the marginal distribution of x? List the values with their probabilities.

(b) What is the marginal distribution of y? List the values with their probabilities.

(c) Calculate $E(x)$. Show your work.

(d) Denote a "centered" version of x by $x_c = x - E(x) = x - \mu_x$.

i. What is the probability distribution of x_c? Give the values with their probabilities.

ii. What is $E(x_c)$? Show your work.

iii. What is the probability distribution of x_c^2? Give the values with their probabilities.

iv. What is $E(x_c^2)$? Show your work.

(e) What is $Var(x)$? If you have been paying attention, you don't have to show any work.

(f) Calculate $E(y)$. Show your work.

(g) Calculate $Var(y)$. Show your work. You may use Question 5a if you wish.

(h) Let $z_1 = g_1(x, y) = x + y$. What is the probability distribution of z_1? Show some work.

(i) Calculate $E(z_1)$. Show your work.

(j) Do we have $E(x + y) = E(x) + E(y)$? Answer Yes or No. Note that the answer *does not require independence*, or even zero covariance.

(k) Let $z_2 = g_2(x, y) = xy$. What is the probability distribution of Z_2? List the values with their probabilities. Show some work.

(l) Calculate $E(z_2)$. Show your work.

(m) Do we have $E(xy) = E(x)E(y)$? Answer Yes or No.

(n) Using the well-known formula of Question 5b, what is $Cov(x, y)$?

(o) Are x and y independent? Answer Yes or No and show some work.

4. Let x_1 and x_2 be continuous random variables that are *independent*. Using the expression for $E(g(\mathbf{x}))$ at the beginning of this assignment, show $E(x_1 x_2) = E(x_1)E(x_2)$. Draw an arrow to the place in your answer where you use independence, and write "This is where I use independence." Because x_1 and x_2 are continuous, you will need to integrate. Does your proof still apply if x_1 and x_2 are discrete?

5. Using the definitions of variance and covariance along with the linear property $E(\sum_{i=1}^{n} a_i y_i) = \sum_{i=1}^{n} a_i E(y_i)$ (no integrals), show the following:

   (a) $Var(y) = E(y^2) - \mu_y^2$

   (b) $Cov(x, y) = E(xy) - E(x)E(y)$

   (c) If x and y are independent, $Cov(x, y) = 0$. Of course you may use Problem 4.

6. Let x be a random variable and let a be a constant. Show

   (a) $Var(ax) = a^2 Var(x)$.

   (b) $Var(x + a) = Var(x)$.

7. Show $Var(x + y) = Var(x) + Var(y) + 2Cov(x, y)$.

8. Let x and y be random variables, and let a and b be constants. Show $Cov(x+a, y+b) = Cov(x, y)$.

9. Let x and y be random variables, with $E(x) = \mu_x$, $E(y) = \mu_y$, $Var(x) = \sigma_x^2$, $Var(y) = \sigma_y^2$, $Cov(x, y) = \sigma_{xy}$ and $Corr(x, y) = \rho_{xy}$. Let a and b be non-zero constants.

   (a) Find $Cov(ax, by)$.

   (b) Find $Corr(ax, by)$. Do not forget that a and b could be negative.

10. Let x_1 and x_2 be discrete random variables. Using the formula for $E(g(\mathbf{x}))$ (note $\mathbf{x}$ is a vector), prove $E(x_1 + x_2) = E(x_1) + E(x_2)$. If you assume independence you get a zero. Does your proof still apply if x_1 and x_2 are continuous?

11. Let $y_1, \ldots, y_n$ be independent random variables with $E(y_i) = \mu$ and $Var(y_i) = \sigma^2$ for $i = 1, \ldots, n$. For this question, please use definitions and familiar properties of expected value, not integrals.

    (a) Find $E(\sum_{i=1}^{n} y_i)$. Are you using independence?

    (b) Find $Var\left(\sum_{i=1}^{n} y_i\right)$. What earlier questions are you using in connection with independence?

    (c) Using your answer to the last question, find $Var(\overline{y})$.

    (d) A statistic T is an *unbiased estimator* of a parameter θ if $E(T) = \theta$. Show that $\overline{y}$ is an unbiased estimator of μ.

    (e) Let $a_1, \ldots, a_n$ be constants and define the linear combination L by $L = \sum_{i=1}^{n} a_i y_i$. Show that if $\sum_{i=1}^{n} a_i = 1$, then L is an unbiased estimator of μ.

    (f) Is $\overline{y}$ a special case of L? If so, what are the a_i values?

    (g) What is $Var(L)$ for general L?

12. Let $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & -4 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ -1 & 3 \end{pmatrix}$ be matrices of constants. Which of the following are possible to compute? Don't do the calculations. Just answer each one Yes or No.

$$\begin{array}{lll} (a)\ \mathbf{A}^{-1} & (b)\ |\mathbf{B}| & (c)\ \mathbf{A}+\mathbf{B} \\ (d)\ \mathbf{A}-\mathbf{B} & (e)\ \mathbf{AB} & (f)\ \mathbf{BA} \\ (g)\ \mathbf{A}'\mathbf{B} & (h)\ \mathbf{B}'\mathbf{A} & (i)\ \mathbf{A}/\mathbf{B} \end{array}$$

13. For the matrices of Question 12, calculate $\mathbf{A}'\mathbf{B}$. My answer is $\mathbf{A}'\mathbf{B} = \begin{pmatrix} 4 & 3 \\ -6 & -3 \end{pmatrix}$.

14. Let $\mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Verify that $\mathbf{c}'\mathbf{d} = 4$ and $\mathbf{cd}' = \begin{pmatrix} 2 & 4 & -2 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

15. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$
    (a) Calculate $\mathbf{AB}$ and $\mathbf{AC}$
    (b) Do we have $\mathbf{AB} = \mathbf{AC}$? Answer Yes or No.
    (c) Prove $\mathbf{B} = \mathbf{C}$. Show your work.

16. Matrix multiplication does not commute. That is, if $\mathbf{A}$ and $\mathbf{B}$ are matrices, in general it is *not* true that $\mathbf{AB} = \mathbf{BA}$ unless both matrices are 1×1. Establish this important fact by making up a simple numerical example in which $\mathbf{A}$ and $\mathbf{B}$ are both 2×2 matrices. Carry out the multiplication, showing $\mathbf{AB} \neq \mathbf{BA}$.

17. Let $\mathbf{A}$ be a square matrix with the determinant of $\mathbf{A}$ (denoted $|\mathbf{A}|$) equal to zero. What does this tell you about $\mathbf{A}^{-1}$? No proof is required here.

18. Recall that $\mathbf{A}$ symmetric means $\mathbf{A} = \mathbf{A}'$. Let $\mathbf{X}$ be an n by p matrix. Prove that $\mathbf{X}'\mathbf{X}$ is symmetric.

19. Let $\mathbf{X}$ be an n by p matrix with $n \neq p$. Why is it incorrect to say that $(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{X}^{-1}\mathbf{X}'^{-1}$?
