Name Key

Student Number

STA 302 f2015 Test 2

For most of the questions on this test, *you have more room than you need.* Don't try to fill up the paper.

1. (5 Points) For the general linear regression model with the columns of $\mathbf{X}$ linearly independent, show either that $\mathbf{X}'\hat{\boldsymbol{\epsilon}} = \mathbf{X}'\boldsymbol{\epsilon}$, or that $\mathbf{X}'\hat{\boldsymbol{\epsilon}} = \mathbf{0}$. One of the statements is true and the other statement is false. Choose the true statement and prove it.

$$X'\hat{\epsilon} = X'(y - \hat{y}) = X'y - X'X\hat{\beta}$$

$$= X'y - \underbrace{X'X(X'X)^{-1}}_{I}X'y = X'y - X'y = 0$$

2. For the general linear regression model with the columns of $\mathbf{X}$ linearly independent,

(a) (15 Points) Show that $Q(\boldsymbol{\beta}) = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = \hat{\boldsymbol{\epsilon}}'\hat{\boldsymbol{\epsilon}} + (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})'(\mathbf{X}'\mathbf{X})(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})$.

$$(y - X\beta)'(y - X\beta) = (y - \hat{y} + \hat{y} - X\beta)'(y - \hat{y} + \hat{y} - X\beta)$$

$$= (\hat{\varepsilon} + \hat{y} - X\beta)'(\hat{\varepsilon} + \hat{y} - X\beta)$$

$$= \hat{\varepsilon}'\hat{\varepsilon} + \hat{\varepsilon}'(\hat{y} - X\beta) + (\hat{y} - X\beta)'\hat{\varepsilon} + (\hat{y} - X\beta)'(\hat{y} - X\beta)$$

$$= \hat{\varepsilon}'\hat{\varepsilon} + \hat{\varepsilon}'(X\hat{\beta} - X\beta) + (X\hat{\beta} - X\beta)'\hat{\varepsilon} + (X\hat{\beta} - X\hat{\beta})'(X\hat{\beta} - X\beta)$$

$$= \hat{\varepsilon}'\hat{\varepsilon} + \underbrace{\hat{\varepsilon}'X}(\hat{\beta} - \beta) + (\hat{\beta} - \beta)'\underbrace{X'\hat{\varepsilon}} + (X(\hat{\beta} - \beta))'X(\hat{\beta} - \beta)$$

zero by Prob 1

$$= \hat{\varepsilon}'\hat{\varepsilon} + (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta)$$

$$= \hat{\varepsilon}'\hat{\varepsilon} + (\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta) \quad \blacksquare$$

(b) (2 Points) Continuing with Question 2, how do you know that the second term of $Q(\boldsymbol{\beta})$ cannot be negative?

Because $X'X$ is positive definite

(c) (3 Points) It's very important that the second term cannot be negative. Why?

Because then there is a minimum at $\beta = \hat{\beta}$

3. (10 Points) For the general linear regression model with the columns of $\mathbf{X}$ linearly independent, you know (and do not need to prove) that $\mathbf{X}'\mathbf{X}$ is positive definite. Also, recall that the square matrix $\mathbf{A}$ is said to have an eigenvalue λ and corresponding eigenvector $\mathbf{v} \neq \mathbf{0}$ if $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.

Prove either (a) The eigenvalues of $\mathbf{X}'\mathbf{X}$ are all positive, or (b) The eigenvalues of $\mathbf{X}'\mathbf{X}$ are all equal to zero. One of the statements is true, and the other is false. Choose the true statement and prove it. *For full marks, show all the steps.*

$$X'Xv = \lambda v$$

$$\Rightarrow v'X'Xv = v'\lambda v = \lambda v'v > 0$$

Since $X'X$ is positive definite and $v \neq 0$

$$\Rightarrow \frac{\lambda v'v}{v'v} = \lambda > \frac{0}{v'v} = 0$$

And eigenvalue is positive.

4. (10 Points) Let the $p \times 1$ random vector $\mathbf{y}$ have expected value $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, and let $\mathbf{A}$ be an $m \times p$ matrix of constants. Starting with the definition of a variance-covariance matrix on the formula sheet, prove $cov(\mathbf{Ay}) = \mathbf{A\Sigma A}'$. You are proving something on the formula sheet, so don't use what you are proving.

$$
\begin{aligned}
Cov(Ay) &= E\{(Ay - A\mu)(Ay - A\mu)'\} \\
&= E\{A(y-\mu)(A(y-\mu))'\} \\
&= E\{A(y-\mu)(y-\mu)'A'\} \\
&= A\,E\{(y-\mu)(y-\mu)'\}A' \\
&= A\Sigma A'
\end{aligned}
$$

5. The most natural choice for estimating the linear combination $\mathbf{a}'\boldsymbol{\beta}$ is the (scalar) statistic $L_0 = \mathbf{a}'\widehat{\boldsymbol{\beta}}$.

(a) (10 Points) Is L_0 an unbiased estimator of $\mathbf{a}'\boldsymbol{\beta}$? Do the calculation, and then write "Yes, unbiased," or "No, biased."

$$E(L_0) = E(a'\hat{\beta}) = a'E(\hat{\beta}) = a'E((X'X)^{-1}X'y)$$

$$= a'(X'X)^{-1}X'E(y) = a'\underbrace{(X'X)^{-1}X'X}_{I}\beta = a'\beta$$

Yes, unbiased

(b) (15 Points) The Gauss-Markov Theorem says that the variance of L_0 is smaller than that of other linear unbiased estimators. What is $Var(L_0)$? Show your work. *For full marks, simplify.* **Circle your final answer.**

$$Var(L_0) = cov(L_0) = Cov(a'\hat{\beta}) = a'cov(\hat{\beta})a$$

$$= a'cov((X'X)^{-1}X'y)a = a'(X'X)^{-1}X'cov(y)X(X'X)^{-1}a$$

$$= a'(X'X)^{-1}X'\sigma^2 I X(X'X)^{-1}a$$

$$= \sigma^2 a'\underbrace{(X'X)^{-1}X'X}_{I}(X'X)^{-1}a$$

$$= \sigma^2 a'(X'X)^{-1}a$$

6. (15 Points) The simple linear regression model is $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ for $i = 1, \ldots, n$, where $\epsilon_1, \ldots, \epsilon_n$ are a random sample from a distribution with expected value zero and variance σ^2. The numbers $x_1, \ldots, x_n$ are known, observed constants, while the parameters β_0 β_1 and σ^2 are unknown constants (parameters). Naturally, the Gauss-Markov Theorem applies in this simple setting.

Let $L = \sum_{i=1}^n c_i Y_i$ be a linear combination of the Y_i values. L is an unbiased estimator of $\mathbf{a}'\boldsymbol{\beta}$, meaning $E(L) = a_1\beta_0 + a_2\beta_1$.

A critical part of the proof of Gauss-Markov is that $\mathbf{a} = \mathbf{X}'\mathbf{c}$. What does this statement mean for the simple linear regression model? The answer is *two equations* involving the x_i, c_i and a_j values. Write these equations in *scalar form.* **Circle both equations.**

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \text{ so}$$

$$a_1 = \sum_{i=1}^n c_i \quad \text{and}$$

$$a_2 = \sum_{i=1}^n c_i x_i$$

7. (15 Points) Let $\mathbf{y}_1 \sim N_p(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ and $\mathbf{y}_2 \sim N_p(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$, where $\mathbf{y}_1$ and $\mathbf{y}_2$ are *independent.* Using moment-generating functions, find the distribution of $\mathbf{s} = \mathbf{y}_1 - \mathbf{y}_2$. For full marks, *clearly indicate where you use independence.* In $\mathbf{s} = \mathbf{y}_1 - \mathbf{y}_2$, notice that it's a minus and not a plus. Finish your answer with a **clear statement of the distribution** of the random vector $\mathbf{s}$.

$$M_s(t) = E(e^{t's}) = E(e^{t'(y_1 - y_2)})$$

$$= E(e^{t'y_1} e^{(-t)'y_2}) \overset{\text{ind}}{=} E(e^{t'y_1}) E(e^{(-t)'y_2})$$

$$= M_{y_1}(t) \, M_{y_2}(-t)$$

$$= e^{t'\mu_1 + \frac{1}{2} t'\Sigma_1 t} \; e^{-t'\mu_2 + \frac{1}{2}(-t)'\Sigma_2(-t)}$$

$$= e^{t'(\mu_1 - \mu_2) + \frac{1}{2} t'(\Sigma_1 + \Sigma_2) t}$$

MGF of $N(\mu_1 - \mu_2, \Sigma_1 + \Sigma_2)$

 Total Marks = 100 points