Name Jerry

Student Number

STA 302f 2015 Test 1B

1. (15 Points) Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$, and let $\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$. Calculate $\mathbf{x}'\mathbf{A}$. Show at least one intermediate step and **circle your final answer**.

$$x'A = (3 \quad 0 \quad 1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$= (3, -2, 1)$$

2. (30 Points) This simple regression model has an unknown intercept, but the slope is fixed at one. Let $Y_i = \beta_0 + x_i + \epsilon_i$ for $i = 1, \ldots, n$, where $\epsilon_1, \ldots, \epsilon_n$ are a random sample (that is, independent and independently distributed) from a distribution with expected value zero and variance σ^2, and β_0 and σ^2 are unknown constants. The numbers $x_1, \ldots, x_n$ are known, observed constants.

(a) Find the least squares estimate of β_0 by minimizing the function

$$Q(\beta_0) = \sum_{i=1}^{n} (Y_i - \beta_0 - x_i)^2$$

over all values of β_0. Let $\widehat{\beta}_0$ denote the point at which $Q(\beta_0)$ is minimal. Show your work. **Circle your final answer.** You need not bother with the second derivative test.

$$\frac{dQ}{d\beta_0} = \frac{d}{d\beta_0} \sum_{i=1}^{n} (Y_i - \beta_0 - x_i)^2 = \sum_{i=1}^{n} \frac{d}{d\beta_0} (Y_i - \beta_0 - x_i)^2$$

$$= \sum_{i=1}^{n} 2(Y_i - \beta_0 - x_i)(-1) \stackrel{\text{set}}{=} 0$$

$$\Rightarrow \sum_{i=1}^{n} Y_i - n\beta_0 - \sum x_i = 0$$

$$\Rightarrow n\beta_0 = \sum_{i=1}^{n} Y_i - \sum_{i=1}^{n} x_i \Rightarrow \beta_0 = \bar{Y} - \bar{x} \text{, and}$$

$$\hat{\beta}_0 = \bar{Y} - \bar{x}$$

(b) Calculate $\widehat{\beta}_0$ for the following data set. The answer is a number. Show just a little work. **Circle your answer.**

x	4	6	7	5	3
y	6	4	2	2	1

$\bar{x} = 25/5 = 5$

$\bar{Y} = 15/5 = 3$

$$\hat{\beta} = \bar{Y} - \bar{x} = 3 - 5 = -2$$

(c) Continuing with Question 2, recall that an estimator is said to be *unbiased* if its expected value equals the parameter it is estimating. Is your $\widehat{\beta}_0$ an unbiased estimator? Show the calculation in detail, and then write either the words "Yes, unbiased" or the words "No, biased." **Circle the words.** *In this question, you are allowed to use any properties of expected value you know, without proof — even if they are not on the formula sheet.*

$$E(\hat{\beta}_0) = E(\bar{Y} - \bar{x}) = E(\bar{Y}) - \bar{x}$$

$$= E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) - \bar{x} = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) - \bar{x}$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(\beta_0 + x_i + \epsilon_i) - \bar{x}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(\beta_0 + x_i + E(\epsilon_i)\right) - \bar{x}$$

$$= \frac{1}{n}\left(n\beta_0 + \Sigma x_i + 0\right) - \bar{x}$$

$$= \beta_0 + \bar{x} - \bar{x} = \beta_0$$

Yes, unbiased

(d) Calculate $Var(\widehat{\beta}_0)$. Show your work. **Circle your answer.** *In this question, you are allowed to use any properties of variance and covariance value you happen to know, without proof — even if they are not on the formula sheet.*

$$Var(\hat{\beta}_0) = Var(\bar{Y} - \bar{x}) = Var(\bar{Y})$$

$$= Var\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) \overset{ind}{=} \frac{1}{n^2}\sum_{i=1}^{n} Var(Y_i)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

3. (10 Points) Let $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ be $p \times 1$ vectors of constants, with $3\mathbf{x}_1 + 5\mathbf{x}_2 = \mathbf{x}_3$. Prove that this set of vectors is linearly dependent. *Use the definition on the formula sheet.*

$$\underset{a_1}{3}\,\underset{\sim}{x}_1 + \underset{a_2}{5}\,\underset{\sim}{x}_2 + \underset{a_3}{(-1)}\,\underset{\sim}{x}_3 = \underset{\sim}{0}$$

a_i are not all zero.

4. (15 Points) Let $\mathbf{A}$ and $\mathbf{B}$ be non-singular matrices, meaning that their inverses exist. Let $\mathbf{C} = \mathbf{AB}$. Do one of these two things. Either

   (a) Give a formula for $\mathbf{C}^{-1}$ and **circle the formula**. Then prove it is the inverse (you have two things to show), or

   (b) Show that $\mathbf{C}^{-1}$ need not exist, by giving a simple numerical example using 2×2 matrices. You may use the fact that if the columns of $\mathbf{C}$ are linearly dependent, it has no inverse.

Pick one and do it. Zero marks if you do both.

$$C^{-1} = B^{-1}A^{-1}$$

(1) $B^{-1}A^{-1}C = B^{-1}\underbrace{A^{-1}A}\,B = B^{-1}I\,B = B^{-1}B = I$

(2) $CB^{-1}A^{-1} = A\underbrace{B\,B^{-1}}_{I}A^{-1} = AA^{-1} = I$

5. (15 Points) Let X_1 and X_2 be continuous random variables that are *independent*. Prove that $E(X_1^2 X_2^2) = E(X_1^2)E(X_2^2)$. Draw an arrow to the place in your answer where you use independence, and write "This is where I use independence." Because X_1 and X_2 are continuous, you will integrate.

$$E(X_1^2 X_2^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2 x_2^2 \, f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

Independence

$$= \int\int x_1^2 x_2^2 \, f_{X_1}(x_1) f_{X_2}(x_2)\, dx_1\, dx_2$$

$$= \int x_2^2 f_{X_2}(x_2) \left[\int x_1^2 f_{X_1}(x_1)\, dx_1 \right] dx_2$$

$$= \int x_2^2 f_{X_2}(x_2)\, E(X_1^2)\, dx_2$$

$$= E(X_1^2) \int x_2^2 f_{X_2}(x_2)\, dx_2$$

$$= E(X_1^2)E(X_2^2)$$

6. (15 Points) Let X_1 be a normal random variable with expected value 1 and variance 3. Let X_2 be a normal random variable with expected value 2 and variance 4. Furthermore, X_1 and X_2 are independent. Let $Y = X_1 + 3X_2$. Use moment-generating functions to find the distribution of Y. **Finish your answer with a clear statement of the distribution, including the numerical values of its parameters.**

$$M_{X_1}(t) = e^{1t + \frac{1}{2} 3t^2}$$

$$M_{X_2}(t) = e^{2t + \frac{1}{2} 4t^2}$$

$$M_Y(t) = M_{X_1+3X_2}(t) \overset{\text{ind}}{=} M_{X_1}(t) \cdot M_{3X_2}(t)$$

$$= M_{X_1}(t)\, M_{X_2}(3t)$$

$$= e^{t + \frac{1}{2}3t^2}\, e^{2(3t) + \frac{1}{2}4(3t)^2}$$

$$= e^{t + \frac{1}{2}3t^2}\, e^{6t + \frac{1}{2}36t^2}$$

$$= e^{7t + \frac{1}{2}39t^2}$$

MGF 8

Normal ($\mu = 7$, $\sigma^2 = 39$)

Cross check

$$E(X_1 + 3X_2) = 1 + 6 = 7$$

$$Var(X_1 + 3X_2) = 3 + 9 \cdot 4 = 39$$

 Total Marks = 100 points