Gauss-Markov Theorem

A clean proof

<u>Thm</u>

Assume the general linear regression model, with the columns of $\underset{\sim}{X}$ linearly independent. Let $\underset{\sim}{a}$ be a $(k+1)\times 1$ vector of constants, and let $\underset{\sim}{c}$ be an $n\times 1$ vector of constants. Let $L = c'Y$, with $E(L) = a'\beta$, so that L is a <u>linear</u> <u>unbiased</u> <u>estimator</u> of $a'\beta$.

Then $Var(L) \geq Var(a'\hat{\beta})$, with equality only if $L = a'\hat{\beta}$

<u>Proof</u>:

$$E(L) = c'X\beta = a'\beta \text{ for all } \beta \in \mathbb{R}^{k+1}, \text{ so that}$$

$$c'X = a' \qquad (*)$$

Note $a'\hat{\beta} = \underbrace{a'(X'X)^{-1}X'}_{c_0'}Y = c_0'Y$.

Now $Var(L) = \sigma^2 c'c = \sigma^2(c-c_0+c_0)'(c-c_0+c_0)$

$$= \sigma^2\left((c-c_0)'(c-c_0) + (c-c_0)'c_0 + c_0'(c-c_0) + c_0'c_0\right)$$

$$\overset{(*)}{=} \sigma^2\left((c-c_0)'(c-c_0) + 0 + 0 + c_0'c_0\right)$$

$$= \sigma^2\left((c-c_0)'(c-c_0) + a'(X'X)^{-1}a\right)$$

The first term is non-negative, and equals zero if and only if $c = c_0$. The second term is free of c. Thus $Var(L)$ is smallest when

$c = c_0 \iff L = a'\hat{\beta}$ ∎