

Random Vectors¹

STA 302 Fall 2015

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Random Vectors and Matrices

See Chapter 3 of *Linear models in statistics* for more detail.

A *random matrix* is just a matrix of random variables. Their joint probability distribution is the distribution of the random matrix. Random matrices with just one column (say, $p \times 1$) may be called *random vectors*.

The expected value of a matrix is defined as the matrix of expected values. Denoting the $p \times c$ random matrix \mathbf{X} by $[X_{i,j}]$,

$$E(\mathbf{X}) = [E(X_{i,j})].$$

Immediately we have natural properties like

$$\begin{aligned} E(\mathbf{X} + \mathbf{Y}) &= E([X_{i,j} + Y_{i,j}]) \\ &= [E(X_{i,j} + Y_{i,j})] \\ &= [E(X_{i,j}) + E(Y_{i,j})] \\ &= [E(X_{i,j})] + [E(Y_{i,j})] \\ &= E(\mathbf{X}) + E(\mathbf{Y}). \end{aligned}$$

Moving a constant matrix through the expected value sign

Let $\mathbf{A} = [a_{i,j}]$ be an $r \times p$ matrix of constants, while \mathbf{X} is still a $p \times c$ random matrix. Then

$$\begin{aligned} E(\mathbf{AX}) &= E\left(\left[\sum_{k=1}^p a_{i,k} X_{k,j}\right]\right) \\ &= \left[E\left(\sum_{k=1}^p a_{i,k} X_{k,j}\right)\right] \\ &= \left[\sum_{k=1}^p a_{i,k} E(X_{k,j})\right] \\ &= \mathbf{A}E(\mathbf{X}). \end{aligned}$$

Similar calculations yield $E(\mathbf{AXB}) = \mathbf{A}E(\mathbf{X})\mathbf{B}$.

Variance-Covariance Matrices

Let \mathbf{X} be a $p \times 1$ random vector with $E(\mathbf{X}) = \boldsymbol{\mu}$. The *variance-covariance matrix* of \mathbf{X} (sometimes just called the *covariance matrix*), denoted by $cov(\mathbf{X})$, is defined as

$$cov(\mathbf{X}) = E \{ (\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})' \} .$$

$$\text{cov}(\mathbf{X}) = E \{ (\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})' \}$$

$$\begin{aligned} \text{cov}(\mathbf{X}) &= E \left\{ \begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \\ X_3 - \mu_3 \end{pmatrix} \begin{pmatrix} X_1 - \mu_1 & X_2 - \mu_2 & X_3 - \mu_3 \end{pmatrix} \right\} \\ &= E \left\{ \begin{pmatrix} (X_1 - \mu_1)^2 & (X_1 - \mu_1)(X_2 - \mu_2) & (X_1 - \mu_1)(X_3 - \mu_3) \\ (X_2 - \mu_2)(X_1 - \mu_1) & (X_2 - \mu_2)^2 & (X_2 - \mu_2)(X_3 - \mu_3) \\ (X_3 - \mu_3)(X_1 - \mu_1) & (X_3 - \mu_3)(X_2 - \mu_2) & (X_3 - \mu_3)^2 \end{pmatrix} \right\} \\ &= \begin{pmatrix} E\{(X_1 - \mu_1)^2\} & E\{(X_1 - \mu_1)(X_2 - \mu_2)\} & E\{(X_1 - \mu_1)(X_3 - \mu_3)\} \\ E\{(X_2 - \mu_2)(X_1 - \mu_1)\} & E\{(X_2 - \mu_2)^2\} & E\{(X_2 - \mu_2)(X_3 - \mu_3)\} \\ E\{(X_3 - \mu_3)(X_1 - \mu_1)\} & E\{(X_3 - \mu_3)(X_2 - \mu_2)\} & E\{(X_3 - \mu_3)^2\} \end{pmatrix} \\ &= \begin{pmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) & \text{Cov}(X_1, X_3) \\ \text{Cov}(X_1, X_2) & \text{Var}(X_2) & \text{Cov}(X_2, X_3) \\ \text{Cov}(X_1, X_3) & \text{Cov}(X_2, X_3) & \text{Var}(X_3) \end{pmatrix}. \end{aligned}$$

So, the covariance matrix $\text{cov}(\mathbf{X})$ is a $p \times p$ symmetric matrix with variances on the main diagonal and covariances on the off-diagonals.

Analogous to $Var(aX) = a^2 Var(X)$

Let \mathbf{X} be a $p \times 1$ random vector with $E(\mathbf{X}) = \boldsymbol{\mu}$ and $cov(\mathbf{X}) = \boldsymbol{\Sigma}$, while $\mathbf{A} = [a_{i,j}]$ is an $r \times p$ matrix of constants. Then

$$\begin{aligned} cov(\mathbf{AX}) &= E\{(\mathbf{AX} - \mathbf{A}\boldsymbol{\mu})(\mathbf{AX} - \mathbf{A}\boldsymbol{\mu})'\} \\ &= E\{\mathbf{A}(\mathbf{X} - \boldsymbol{\mu})(\mathbf{A}(\mathbf{X} - \boldsymbol{\mu}))'\} \\ &= E\{\mathbf{A}(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})'\mathbf{A}'\} \\ &= \mathbf{A}E\{(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})'\}\mathbf{A}' \\ &= \mathbf{A}cov(\mathbf{X})\mathbf{A}' \\ &= \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}' \end{aligned}$$

Positive definite is a natural assumption

For covariance matrices

- $cov(\mathbf{X}) = \Sigma$
- Σ positive definite means $\mathbf{a}'\Sigma\mathbf{a} > 0$. for all $\mathbf{a} \neq \mathbf{0}$.
- $Y = \mathbf{a}'\mathbf{X} = a_1X_1 + \dots + a_pX_p$ is a scalar random variable.
- $Var(Y) = \mathbf{a}'cov(\mathbf{X})\mathbf{a} = \mathbf{a}'\Sigma\mathbf{a}$
- Σ positive definite just says that the variance of any (non-trivial) linear combination is positive.
- This is often what you want (but not always).

Matrix of covariances between two random vectors

Let \mathbf{X} be a $p \times 1$ random vector with $E(\mathbf{X}) = \boldsymbol{\mu}_x$ and let \mathbf{Y} be a $q \times 1$ random vector with $E(\mathbf{Y}) = \boldsymbol{\mu}_y$.

The $p \times q$ matrix of covariances between the elements of \mathbf{X} and the elements of \mathbf{Y} is

$$C(\mathbf{X}, \mathbf{Y}) = E \{ (\mathbf{X} - \boldsymbol{\mu}_x)(\mathbf{Y} - \boldsymbol{\mu}_y)' \}.$$

Adding a constant has no effect

On variances and covariances

It's clear from the definitions

- $cov(\mathbf{X}) = E \{(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})'\}$
- $C(\mathbf{X}, \mathbf{Y}) = E \{(\mathbf{X} - \boldsymbol{\mu}_x)(\mathbf{Y} - \boldsymbol{\mu}_y)'\}$

That

- $cov(\mathbf{X} + \mathbf{a}) = cov(\mathbf{X})$
- $C(\mathbf{X} + \mathbf{a}, \mathbf{Y} + \mathbf{b}) = C(\mathbf{X}, \mathbf{Y})$

For example, $E(\mathbf{X} + \mathbf{a}) = \boldsymbol{\mu} + \mathbf{a}$, so

$$\begin{aligned} cov(\mathbf{X} + \mathbf{a}) &= E \{(\mathbf{X} + \mathbf{a} - (\boldsymbol{\mu} + \mathbf{a}))(\mathbf{X} + \mathbf{a} - (\boldsymbol{\mu} + \mathbf{a}))'\} \\ &= E \{(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})'\} \\ &= cov(\mathbf{X}) \end{aligned}$$

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<http://www.utstat.toronto.edu/~brunner/oldclass/302f15>