# STA 302f15 Assignment Seven

These problems are preparation for the quiz, and are not to be handed in. As usual, **you might be asked to prove things that are not true**. In this case you should say why the statement is not always true.

1. Let the continuous random vectors $\mathbf{y_1}$ and $\mathbf{y_2}$ be independent. Show that their joint moment-generating function is the product of their moment-generating functions. Since $\mathbf{y_1}$ and $\mathbf{y_2}$ are continuous, you will integrate. It is okay to represent a multiple integral with a single integral sign.

2. Show that if $\mathbf{y} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with $\boldsymbol{\Sigma}$ positive definite, then $W = (\mathbf{y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu})$ has a chi-square distribution with $p$ degrees of freedom.

3. Let $Y_1, \ldots, Y_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution.
   (a) Show $Cov(\overline{Y}, (Y_j - \overline{Y})) = 0$ for $j = 1, \ldots, n$.
   (b) Show that $\overline{Y}$ and $S^2$ are independent.
   (c) Show that
   $$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1),$$
   where $S^2 = \frac{\sum_{i=1}^n (Y_i - \overline{Y})^2}{n-1}$. Hint: $\sum_{i=1}^n (Y_i - \mu)^2 = \sum_{i=1}^n \left(Y_i - \overline{Y} + \overline{Y} - \mu\right)^2 = \ldots$

4. Recall the definition of the $t$ distribution. If $Z \sim N(0,1)$, $W \sim \chi^2(\nu)$ and $Z$ and $W$ are independent, then $T = \frac{Z}{\sqrt{W/\nu}}$ is said to have a $t$ distribution with $\nu$ degrees of freedom, and we write $T \sim t(\nu)$. As Question 3, let $Y_1, \ldots, Y_n$ be random sample from a $N(\mu, \sigma^2)$ distribution. Show that $T = \frac{\sqrt{n}(\overline{Y} - \mu)}{S} \sim t(n-1)$.

5. In the multiple linear regression model, let the columns of the $\mathbf{X}$ matrix be linearly independent. Either (a) show that $(\mathbf{X}'\mathbf{X})^{-1/2}$ is symmetric, or (b) show by a simple numerical example that $(\mathbf{X}'\mathbf{X})^{-1/2}$ may not be symmetric.

6. In the general linear regression model with normal error terms, what is the distribution of $\mathbf{y}$?

7. You know that the least squares estimate of $\boldsymbol{\beta}$ is $\widehat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$. What is the distribution of $\widehat{\boldsymbol{\beta}}$ assuming normal error terms? Show the calculations.

8. Let $\widehat{\mathbf{y}} = \mathbf{X}\hat{\boldsymbol{\beta}}$. What is the distribution of $\widehat{\mathbf{y}}$ assuming normal error terms? Show the expected value and covariance matrix calculations.

9. Let the vector of residuals $\hat{\boldsymbol{\epsilon}} = \mathbf{y} - \widehat{\mathbf{y}}$. What is the distribution of $\hat{\boldsymbol{\epsilon}}$ assuming normal error terms? Show the calculations. Simplify both the expected value (which is zero) and the covariance matrix.

10. For the general linear regression model with normal error terms, show that the $n \times (k+1)$ matrix of covariances $C(\hat{\boldsymbol{\epsilon}}, \widehat{\boldsymbol{\beta}}) = \mathbf{0}$. Why does this show that $SSE = \hat{\boldsymbol{\epsilon}}'\hat{\boldsymbol{\epsilon}}$ and $\widehat{\boldsymbol{\beta}}$ are independent?

11. Calculate $C(\widehat{\boldsymbol{\epsilon}}, \widehat{\mathbf{y}})$; show your work. Why should you have known this answer without doing the calculation, assuming normal error terms? Why does the assumption of normality matter?

12. For the general linear regression model with normal error terms, show that $\hat{\boldsymbol{\epsilon}}$ and $\overline{y}$ are independent.

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