STA 302f15 Assignment Four[1]

For this assignment, Chapter 3 in the text contains material on random vectors, and Chapter 6 contains material on simple regression. For this assignment, see just Sections 6.1 and 6.2. You are responsible for what is in this assignment, not everything that's in the text.

This exercise set has an unusual feature. *Some of the questions ask you to prove things that are false.* That is, they are not true in general. In such cases, just write "The statement is false," and give a brief explanation to make it clear that you are not just guessing. The explanation is essential for full marks. A small counter-example is always good enough.

All the problems are preparation for the quiz in tutorial on Friday October 8th, and are not to be handed in.

1. Do problem 3.9 in the text.
2. Let $\mathbf{X} = [X_{ij}]$ be a random matrix. Show $E(\mathbf{X}') = E(\mathbf{X})'$.
3. Let $\mathbf{X}$ be a random matrix, and $\mathbf{B}$ be a matrix of constants. Show $E(\mathbf{XB}) = E(\mathbf{X})\mathbf{B}$. Recall the definition $\mathbf{AB} = [\sum_k a_{i,k}b_{k,j}]$.
4. Let the $p \times 1$ random vector $\mathbf{X}$ have expected value $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, and let $\mathbf{A}$ be an $m \times p$ matrix of constants. Prove that the variance-covariance matrix of $\mathbf{AX}$ is either
   - $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$, or
   - $\mathbf{A}^2\boldsymbol{\Sigma}$..

   Pick one and prove it. Start with the definition of a variance-covariance matrix on the formula sheet.
5. Do problem 3.10 in the text.
6. Let $\mathbf{X}$ be a $p \times 1$ random vector. Starting with the definition on the formula sheet, prove $cov(\mathbf{X}) = \mathbf{0}$..
7. Let the $p \times 1$ random vector $\mathbf{X}$ have mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, let $\mathbf{A}$ be an $r \times p$ matrix of constants, and let $\mathbf{c}$ be an $r \times 1$ vector of constants. Find $cov(\mathbf{AX} + \mathbf{c})$. Show your work.
8. Let the scalar random variable $Y = \mathbf{v}'\mathbf{X}$. What is $Var(Y)$? Use this to prove that *any* variance-covariance matrix must be positive semi-definite.
9. The square matrix $\mathbf{A}$ has an eigenvalue equal to λ with corresponding eigenvector $\mathbf{x} \neq \mathbf{0}$ if $\mathbf{Ax} = \lambda\mathbf{x}$.
   (a) Show that the eigenvalues of a variance-covariance matrix cannot be negative.
   (b) How do you know that the determinant of a variance-covariance matrix must be greater than or equal to zero? The answer is one short sentence.

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(c) Let X and Y be scalar random variables. Recall $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$. Using what you have shown about the determinant, show $-1 \leq Corr(X,Y) \leq 1$.

10. Let $\mathbf{X}$ be a $p \times 1$ random vector with mean $\boldsymbol{\mu}_x$ and variance-covariance matrix $\boldsymbol{\Sigma}_x$, and let $\mathbf{Y}$ be a $q \times 1$ random vector with mean $\boldsymbol{\mu}_y$ and variance-covariance matrix $\boldsymbol{\Sigma}_y$.
    (a) What is the (i,j) element of $cov(\mathbf{X}, \mathbf{Y})$? See the definition on the formula sheet.
    (b) Find an expression for $cov(\mathbf{X} + \mathbf{Y})$ in terms of $\boldsymbol{\Sigma}_x$, $\boldsymbol{\Sigma}_y$ and $cov(\mathbf{X}, \mathbf{Y})$. Show your work.
    (c) Simplify further for the special case where $Cov(X_i, Y_j) = 0$ for all i and j.
    (d) Let $\mathbf{c}$ be a $p \times 1$ vector of constants and $\mathbf{d}$ be a $q \times 1$ vector of constants. Find $cov(\mathbf{X} + \mathbf{c}, \mathbf{Y} + \mathbf{d})$. Show your work.
11. Starting with the definition on the formula sheet, show $cov(\mathbf{X}, \mathbf{Y}) = cov(\mathbf{Y}, \mathbf{X})$..
12. Starting with the definition on the formula sheet, show $cov(\mathbf{X}, \mathbf{Y}) = \mathbf{0}$..
13. Do problem 3.20 in the text. The answer is in the back of the book.
14. Do problem 3.21 in the text. The answer is in the back of the book.

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15. In the textbook, do Problems 6.1, 6.2 and 6.3a.
16. For the simple linear regression model (6.1) in the text,
    (a) What is $E(y_i)$?
    (b) What is $Var(y_i)$?
    (c) What is $Cov(y_i, y_j)$ for $i \neq j$?
    (d) Differentiate to obtain (6.3) and (6.4).
    (e) Prove that $\sum_{i=1}^n \widehat{\epsilon}_i = 0$.
    (f) Prove that the least squares line always goes through the point $(\overline{x}, \overline{y})$.
17. Do problems 6.5 and 6.10 in the text.
