STA 302f15 Assignment Three[1]

For this assignment, Chapter 2 in the text contains material on matrix algebra. You are responsible for what is in this assignment, not everything that's in the text. Questions 2 and 3 are to be done with R. Please print the two sets of R output on separate pieces of paper. You may be asked to hand one of them in, but not the other. Except for the R parts, these problems are preparation for the quiz in tutorial on Friday October 1st, and are not to be handed in.

*Remember that the R parts are **not group projects**. Do the work yourself. Don't help anybody until you are finished. Don't help anybody who has not started yet. Never look at anyone else's code or let anyone look at yours.*

1. In the textbook, do Problems 2.27, 2.28, 2.35, 2.36, 2.37, 2,38 (Prove the "if" part too), 2.53, 2.76.

2. In the textbook, do 2.14 a, e, g, and m using R. Show both input (creation of the matrices) and output. Label the output (give letters) using comments. Bring the printout to the quiz.

3. Make up a your own 4×4 symmetric matrix that is not singular (that is, the inverse exists), and is *not a diagonal matrix*. If your first try is singular, try again. Call it $\mathbf{A}$. Enter it into R using `rbind` (see lecture slides). Make sure to display the input. Then,

   (a) Calculate $|\mathbf{A}^{-1}|$ and $1/|\mathbf{A}|$, verifying that they are equal.

   (b) Calculate $|\mathbf{A}^2|$ and $|\mathbf{A}|^2$, verifying that they are equal.

   (c) Calculate the eigenvalues and eigenvectors of $\mathbf{A}$.

   (d) Calculate $\mathbf{A}^{1/2}$.

   (e) Calculate $\mathbf{A}^{-1/2}$.

   Display both input and output for each part. Label the output with comments. Bring the printout to the quiz.

4. Recall the definition of linear independence. The columns of $\mathbf{X}$ are said to be *linearly dependent* if there exists a $p \times 1$ vector $\mathbf{v} \neq \mathbf{0}$ with $\mathbf{Xv} = \mathbf{0}$. We will say that the columns of $\mathbf{X}$ are linearly *independent* if $\mathbf{Xv} = \mathbf{0}$ implies $\mathbf{v} = \mathbf{0}$. Let $\mathbf{A}$ be a square matrix. Show that if the columns of $\mathbf{A}$ are linearly dependent, $\mathbf{A}^{-1}$ cannot exist. Hint: $\mathbf{v}$ cannot be both zero and not zero at the same time.

5. Let $\mathbf{a}$ be an $n \times 1$ matrix of real constants. How do you know $\mathbf{a}'\mathbf{a} \geq 0$?

6. Recall the *spectral decomposition* of a square symmetric matrix (For example, a variance-covariance matrix). Any such matrix $\mathbf{\Sigma}$ can be written as $\mathbf{\Sigma} = \mathbf{CDC}'$, where $\mathbf{C}$ is a matrix whose columns are the (orthonormal) eigenvectors of $\mathbf{\Sigma}$, $\mathbf{D}$ is a diagonal matrix of the corresponding eigenvalues, and $\mathbf{C}'\mathbf{C} = \mathbf{CC}' = \mathbf{I}$.

   (a) Let $\mathbf{\Sigma}$ be a square symmetric matrix with eigenvalues that are all strictly positive.

      i. What is $\mathbf{D}^{-1}$?

      ii. Show $\mathbf{\Sigma}^{-1} = \mathbf{CD}^{-1}\mathbf{C}'$

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(b) Let $\mathbf{\Sigma}$ be a square symmetric matrix, and this time some of the eigenvalues might be zero.

i. What do you think $\mathbf{D}^{1/2}$ might be?

ii. Define $\mathbf{\Sigma}^{1/2}$ as $\mathbf{C}\mathbf{D}^{1/2}\mathbf{C}'$. Show $\mathbf{\Sigma}^{1/2}$ is symmetric.

iii. Show $\mathbf{\Sigma}^{1/2}\mathbf{\Sigma}^{1/2} = \mathbf{\Sigma}$.

(c) Now return to the situation where the eigenvalues of the square symmetric matrix $\mathbf{\Sigma}$ are all strictly positive. Define $\mathbf{\Sigma}^{-1/2}$ as $\mathbf{C}\mathbf{D}^{-1/2}\mathbf{C}'$, where the elements of the diagonal matrix $\mathbf{D}^{-1/2}$ are the reciprocals of the corresponding elements of $\mathbf{D}^{1/2}$.

i. Show that the inverse of $\mathbf{\Sigma}^{1/2}$ is $\mathbf{\Sigma}^{-1/2}$, justifying the notation.

ii. Show $\mathbf{\Sigma}^{-1/2}\mathbf{\Sigma}^{-1/2} = \mathbf{\Sigma}^{-1}$.

(d) The (square) matrix $\mathbf{\Sigma}$ is said to be *positive definite* if $\mathbf{v}'\mathbf{\Sigma}\mathbf{v} > 0$ for all vectors $\mathbf{v} \neq \mathbf{0}$. Show that the eigenvalues of a positive definite matrix are all strictly positive.

(e) Let $\mathbf{\Sigma}$ be a symmetric, positive definite matrix. Putting together a couple of results you have proved above, establish that $\mathbf{\Sigma}^{-1}$ exists.

7. Using the Spectral Decomposition Theorem and $tr(\mathbf{AB}) = tr(\mathbf{BA})$, prove that the trace is the sum of the eigenvalues for a symmetric matrix.

8. Using the Spectral Decomposition Theorem and $|\mathbf{AB}| = |\mathbf{BA}|$, prove that the determinant of a symmetric matrix is the product of its eigenvalues.
